NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE

No. 969

ON A METHOD OF CONSTRUCTING TWO-DIMENSIONAL SUBSONIC COMPRESSIBLE FLOWS AROUND CLOSED PROFILES

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Washington
March 1945

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SUMMARY

It is shown that under certain conditions a two-dimensional subsonic compressible flow around an airfoil profile can be derived from an incompressible flow around another profile. The connection between these two "conjugate flows" is given by a simple conformal transformation of the respective hodograph planes.

The transformation of a given incompressible flow into a compressible flow around a slightly distorted profile reduces to the integration of a linear partial differential equation in the physical plane of the incompressible flow. An approximate solution of this equation is indicated. Further research is necessary in order to extend the applicability of the method and in order to reduce the computational work involved in the rigorous solution to an acceptable minimum.

The transformation of an incompressible flow into a compressible one can be carried out completely and in a closed form under the assumption of the linearized pressure-density relation. The final formulas represent an extension of the result of Von Kármán and Tsien, to which they reduce in the special case of a flow without circulation. It is shown that essentially all compressible flows can be obtained by this method.

INTRODUCTION

The high level which has been attained by the theory of two-dimensional incompressible flows is due to the fact that this theory is based upon a highly developed mathematical theory, that of analytic functions of a complex variable. Every analytic function yields a possible flow pattern and vice versa. Furthermore, the use of transformations performed by means of analytic functions (conformal transformations) permits the derivation of all possible flows from a few simple standard forms. It seems obvious that the theory of two-dimensional compressible flows (at least as far as subsonic flows are concerned) requires the development of a similar mathematical background.

The theory of sigma-monogenic functions (references 1 and 2) is an attempt to study a class of complex functions the role of which in gas dynamics is comparable to that of analytic functions in the theory of incompressible flows. Gelbart (reference 3) has outlined the application of this method to the study of compressible flows. Further applications depend upon the investigation of singularities of sigma-monogenic functions. (Such an investigation is being conducted.) Reference also is made to a recent report by Garrick and Kaplan (reference 4). The investigation of transformations which for the case of compressible flows take the place of conformal transformations is the main theoretical aim of the present report.

The following remarks may indicate in which way such transformations enter into the study of compressible flows around airfoil profiles.

The differential equations governing the steady two-dimensional potential flow of a compressible fluid are nonlinear and therefore difficult to treat as far as both theoretical considerations and numerical computations are concerned. Molenbroek (reference 5) and Tchaplygin (reference 6) have shown that the equations become linear in the hodograph plane. There exist various methods of obtaining solutions of these hodograph equations, in particular of obtaining solutions which in a certain sense correspond to given solutions of the Cauchy-Riemann equations - that is, to given incompressible flows. Some of these methods are: separation of variables (so successfully used by Tchaplygin in solving jet problems), the method of integral operators (Bergman, reference 7), the method of sigma-monogenic functions (Bers and Gelbart, reference 1), and an approximate method of Temple and Yarwood (reference 8). Furthermore, by modifying the pressure-density relation, the hodograph equations can be made to coincide with the Cauchy-Riemann equations, as is done by Tchaplygin (reference 6), Busemann (reference 9), Demtchenko (reference 10), and, in a more rational way, by Von Kármán and Tsien (references 11 and 12).

However, the real difficulty lies not only in obtaining a solution but in obtaining the "right" solution - that is, one which leads to a flow of a desired type in the physical plane, for example, to a flow around a closed profile. This difficulty is illustrated by the fact that even for the case when the hodograph equations are simply the Cauchy-Riemann equations, the computation of flows around closed profiles has until now been carried out only for a special case (flows without circulation).

Therefore the study of the flow in the hodograph plane must be supplemented by the investigation of the mapping of the physical plane into the hodograph plane and of possible transformations of incompressible flows around closed profiles into compressible flows of the same type. The present report is an attempt in this direction.

The methods outlined in this report are at present restricted to flows which are everywhere subsonic. Flows of mixed type (subsonic main flow with locally supersonic regions) are of more interest from the theoretical as well as from the practical viewpoint. It is thought, however, that the solution of the problem of entirely subsonic flows is a necessary prerequisite for a successful theoretical treatment of the much more difficult problem of mixed flows.

This investigation, carried out at Brown University, was sponsored by and conducted with the financial assistance of the National Advisory Committee for Aeronautics. The author is indebted to Mr. J. B. Diaz for valuable assistance.

SYMBOLS

a	speed of sound
ds	line element in the z-plane; line element of the profile P
dS	non-Euclidean length of ds
$d\sigma$	line element in the ζ-plane; line element of the profile Π
$E(P)$	domain exterior to the profile P
$E(\Pi)$	domain exterior to the profile Π
$\exp(\ )$	exponential function of $(\ ) = e^{(\ )}$
$G(\zeta)$	complex potential of an incompressible flow in the ζ-plane, normalized so that $G'(\infty) = 1$
H	hodograph of a compressible flow in the z-plane
H^*	distorted hodograph of a compressible flow in the z-plane
i	imaginary unit; subscript referring to an incompressible flow
$\mathrm{Im}(\ )$	imaginary part of $(\ )$
K	bound for the ratio of maximal speed to stream speed in the conjugate incompressible flow
M	Mach number
M_∞	stream Mach number
n	modulus of the correspondence between two flows
o	subscript referring to the state of fluid at rest
$o(R)$	a function f such that $\lvert f/R\rvert \longrightarrow 0$ as $R \rightarrow R_1$, R_1 being some specified limit

$O(R)$ a function f such that $|f/R|$ remains bounded as $R \rightarrow R_1$, R_1 being some specified limit

p pressure

P profile in the z-plane

q speed

q^* distorted speed

q_i speed of an incompressible flow

$\tilde{q}$ speed of a fictitious compressible flow in the ζ-plane

R positive constant

R_P radius of curvature of the profile P

$R_\sqcap$ radius of curvature of the profile $\sqcap$

$Re(\;)$ real part of $(\;)$

T coefficient of the symmetrized hodograph equations

u, v components of the velocity

u^*, v^* components of the distorted velocity

x, y Cartesian coordinates in the z-plane

X, Y Cartesian coordinates in the Z-plane

$z = x + iy$ complex variable in the physical plane of the compressible flow

$Z = X + iY$ auxiliary complex variable

γ exponent in the polytropic relation

$\zeta = \xi + i\eta$ complex variable in the plane of the incompressible flow

θ angle between the velocity vector of the compressible flow and the x-axis

θ_1 angle between the velocity vector of the incompressible flow and the ξ-axis

λ logarithm of the distorted speed

μ value of the distorted speed at infinity for $\gamma = -1$

ξ, η Cartesian coordinates in the ζ-plane

Π profile in the ζ-plane

ρ density

$\tilde{\rho}$ density of the fictitious compressible flow in the ζ-plane

φ velocity potential

χ angle between a line element ds and a streamline

ψ stream function

Ω complex potential of the conjugate incompressible flow

Ω_1 complex potential of an incompressible flow around the circle $|Z| = R$

∞ the point infinity; subscript referring to the state of the fluid at infinity

$\overline{(\ )}$ complex conjugate of ( )

$|(\ )|$ absolute value of ( )

The units are chosen so that ρ_0 (stagnation density) and a_0 (speed of sound at a stagnation point) are both equal to unity.

ANALYSIS

I. GENERAL CONCEPTS

Fundamental Relations

It will be assumed that in a compressible fluid the pressure p is a given increasing function of the density ρ. The velocity of sound a is given by

$$a^2 = \frac{dp}{d\rho}$$

If the flow is irrotational, it follows from Bernoulli's equation

$$\frac{q^2}{2} + \int_{p_0}^{p} \frac{dp}{\rho} = 0$$

that the density is a given function of the speed q. Since the preceding equation can be written in the form

$$q\,dq + a^2 \frac{d\rho}{\rho} = 0$$

the Mach number $M = q/a$ is given by the relation

$$M^2 = -\frac{q}{\rho}\frac{d\rho}{dq}$$

The units will be chosen so that

$$a_0 = 1, \quad \rho_0 = 1 \tag{1}$$

(the subscript o referring to the state of the fluid at rest). This is equivalent to the introduction of <u>dimensionless variables</u> q/a_0, ρ/ρ_0.

The pressure-density relation used in gas dynamics is of the form

$$p = A + B\rho^{\gamma} \tag{2}$$

This relation includes the case of an <u>isothermal</u> flow, where

$$\gamma = 1 \tag{3}$$

and that of an <u>adiabatic</u> flow with

$$1 < \gamma < 1.66 \tag{4}$$

(The standard value of γ for air is 1.405.) The value $\gamma = 2$ corresponds to the analogy between a two-dimensional gas flow and a flow of water in an open channel. (Cf., for instance, Von Kármán, reference 12.) In the foregoing cases

$$A = 0, \quad B = p_c$$

The differential equations of a potential gas flow are considerably simplified by introducing the <u>linearized pressure-volume relation</u> with

$$\gamma = -1 \tag{5}$$

and

$$A = p_\infty + a_\infty^2 \rho_\infty, \quad B = -a_\infty^2 \rho_\infty^2$$

where the subscript ∞ refers to the state of the fluid at infinity, and a_∞, ρ_∞, p_∞ have been determined according to the actual pressure-density relation (with $A = 0$, $B = p_0$, $\gamma > 1$). Using this relation amounts to replacing the curve giving the actual pressure-density relation in the $(1/\rho, p)$-plane by its tangent at the point $(1/\rho_\infty, p_\infty)$. The linearized pressure-density relation has been introduced by Von Kármán and Tsien (references 11 and 12), (and formerly in a less general form by Tchaplygin (reference 6), Demtchenko (reference 10), and Buseman (reference 9).

The relations between ρ, M, and q obtained from (2) depend only upon γ (and not upon A and B). For $\gamma=1$

$$\rho = e^{-q^2/2} \tag{6}$$

$$M = q \tag{7}$$

for $\gamma \neq 1$

$$\rho = \left(1 - \frac{\gamma - 1}{2} q^2\right)^{1/(\gamma-1)} \tag{8}$$

$$M^2 = \frac{q^2}{1 - \frac{\gamma - 1}{2} q^2} \tag{9}$$

so that for $\gamma = -1$

$$\rho = \frac{1}{\sqrt{1 + q^2}} \tag{10}$$

$$M^2 = \frac{q^2}{1 + q^2} \tag{11}$$

and

$$\rho^2 = 1 - M^2 \tag{12}$$

In figure 1, ρ is plotted as a function of q (for $\gamma = -1$, 1, 1.405).

For $\gamma = -1$, the flow is always subsonic. For $\gamma \neq 1$, the flow is subsonic as long as

$$q < q_s = \sqrt{\frac{2}{\gamma + 1}} \tag{13}$$

It will be convenient to use the <u>distorted speed</u> q^* (first introduced by Busemann (reference 9))

$$q^* = \exp \int^{q} \frac{\sqrt{1 - M^2}}{q}\, dq \tag{14}$$

$$q^* = 1 \quad \text{for} \quad M = 1 \tag{15}$$

For $\gamma = 1$

$$q^* = \frac{q}{1 + \sqrt{1 - q^2}} e^{\sqrt{1-q^2}} \tag{16}$$

for $\gamma > 1$

$$q^* = \left\{ \frac{1 - \sqrt{1 - M^2}}{1 + \sqrt{1 - M^2}} \left[\frac{1 + \sqrt{(\gamma - 1)/(\gamma + 1)}\sqrt{1 - M^2}}{1 - \sqrt{(\gamma - 1)/(\gamma + 1)}\sqrt{1 - M^2}} \right]^{\sqrt{\frac{\gamma+1}{\gamma-1}}} \right\}^{\frac{1}{2}} \tag{17}$$

and for $\gamma = -1$

$$q^* = \frac{q}{1 + \sqrt{1 + q^2}} \tag{18}$$

(In fig. 2, q and q^* are plotted as functions of M for $\gamma = 1.405$.)

For an incompressible flow ρ is constant, $M = 0$, and $q^* = q$.

Equations of Motion

The x and y components of the dimensionless velocity of a two-dimensional potential steady gas flow, u and v, satisfy the condition of irrotationality

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$$

and the continuity equation

$$\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0$$

These equations imply the existence of a velocity potential $\varphi(x,y)$ and of a stream function $\psi(x,y)$ so that

$$u = \frac{\partial \varphi}{\partial x}, \quad v = \frac{\partial \varphi}{\partial y} \tag{19}$$

and

$$\rho v = -\frac{\partial \psi}{\partial x}, \quad \rho u = \frac{\partial \psi}{\partial y} \tag{20}$$

It follows that

$$\left.\begin{aligned} \frac{\partial \varphi}{\partial x} &= \frac{1}{\rho}\frac{\partial \psi}{\partial y} \\ \frac{\partial \varphi}{\partial y} &= -\frac{1}{\rho}\frac{\partial \psi}{\partial x} \end{aligned}\right\} \tag{21}$$

The elimination of either ψ or φ leads to the second-order equations

$$\frac{\partial}{\partial x}\left(\rho \frac{\partial \varphi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\rho \frac{\partial \varphi}{\partial y}\right) = 0 \tag{22}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{\rho} \frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\rho} \frac{\partial \psi}{\partial y}\right) = 0 \tag{23}$$

The fundamental equations (21) to (23) are of a purely kinematic nature and hold independently of the equation of state. If the density ρ is considered as a given function of space $(\rho = \rho(x,y))$, the equations (22) are linear and always of the elliptic type, no matter whether the flow is subsonic or supersonic.

However, the important case is that in which the density is a given function of pressure and therefore also a given function of the magnitude of the (dimensionless) velocity $q = \sqrt{u^2 + v^2}$. (Cf. preceding sec.) In this case equations (21) to (23) are non-linear (more precisely: quasi-linear).

The velocity distribution in a given flow is uniquely determined by the boundary conditions and by the functional relation

$$\rho = f(q), \quad q = \sqrt{\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2}$$

This remark justifies the use of the relation $\gamma = -1$ for subsonic flows of not too high Mach number. For, replacing the actual value of γ by -1 changes the function $\rho = f(q)$ (and the differential equation for φ), $q < 1$, only slightly. (Cf. fig. 1.) From the equations given in the preceding section, it follows that the change of ρ is of the order of magnitude

$$\frac{\gamma + 1}{8} M^4$$

Molenbroek (reference 5) and Tčhaplygin (reference 6) showed that linear equations can be obtained by considering φ and ψ as functions of q and θ, where θ is the angle between the velocity vector and the x-axis:

$$\theta = (\tan^{-1}) \frac{v}{u} \tag{24}$$

The equations take the form

$$\left.\begin{aligned} \frac{\partial\varphi}{\partial\theta} &= \frac{q}{\rho}\frac{\partial\psi}{\partial q} \\ \frac{\partial\varphi}{\partial q} &= -\frac{1}{\rho q}(1 - M^2)\frac{\partial\psi}{\partial\theta} \end{aligned}\right\} \tag{25}$$

These equations can be brought to a symmetric form by replacing the independent variable q by

$$\lambda = \log q^*$$

q^* being the distorted speed. By virtue of (14), (25) can be written in the form

$$\left.\begin{aligned} \frac{\partial\varphi}{\partial\theta} &= T\frac{\partial\psi}{\partial\lambda} \\ \frac{\partial\varphi}{\partial\lambda} &= -T\frac{\partial\psi}{\partial\theta} \end{aligned}\right\} \tag{26}$$

with

$$T = \frac{\sqrt{1 - M^2}}{\rho} \tag{27}$$

For the isothermal case ($\gamma = 1$)

$$T = \sqrt{1 - q^2}\, e^{\frac{q^2}{2}} \tag{28}$$

For the polytropic case ($\gamma > 1$)

$$T = \frac{\left(1 - \frac{\gamma + 1}{2} q^2\right)^{\frac{1}{2}}}{\left(1 - \frac{\gamma - 1}{2} q^2\right)^{\frac{1}{2}(\gamma+1)/(\gamma-1)}} \tag{29}$$

For the case of the linearized equation of state ($\gamma = -1$)

$$T \equiv 1 \tag{30}$$

and equations (26) are Cauchy-Riemann equations. In figure 2, T is plotted as a function of the local Mach number M (for $\gamma = 1.405$). It should be noted that T is a known function of q and therefore also of q^*.

The main advantage of the symmetric form (26) consists in the fact that the symmetric equations are invariant under conformal transformation of the θ, λ-plane: If new independent variables ξ and η are introduced by setting

$$\xi + i\eta = F(\theta + i\lambda)$$

F being an analytic function, then

$$\left.\begin{aligned} \frac{\partial \varphi}{\partial \xi} &= T \frac{\partial \psi}{\partial \eta} \\ \frac{\partial \varphi}{\partial \eta} &= -T \frac{\partial \psi}{\partial \xi} \end{aligned}\right\} \tag{31}$$

with

$$T = T\left\{q\left[q^*(\xi,\eta)\right]\right\}$$

In fact, (26) can be written in the form

$$\frac{\partial\varphi}{\partial\xi}\frac{\partial\xi}{\partial\theta} + \frac{\partial\varphi}{\partial\eta}\frac{\partial\eta}{\partial\theta} = T\left\{\frac{\partial\psi}{\partial\xi}\frac{\partial\xi}{\partial\lambda} + \frac{\partial\psi}{\partial\eta}\frac{\partial\eta}{\partial\lambda}\right\}$$

$$\frac{\partial\varphi}{\partial\xi}\frac{\partial\xi}{\partial\lambda} + \frac{\partial\varphi}{\partial\eta}\frac{\partial\eta}{\partial\lambda} = -T\left\{\frac{\partial\psi}{\partial\xi}\frac{\partial\xi}{\partial\theta} + \frac{\partial\psi}{\partial\eta}\frac{\partial\eta}{\partial\theta}\right\}$$

Eliminating the derivatives of ξ and η by means of the Cauchy-Riemann equations

$$\frac{\partial\xi}{\partial\theta} = \frac{\partial\eta}{\partial\lambda},\quad \frac{\partial\xi}{\partial\lambda} = -\frac{\partial\eta}{\partial\theta}$$

yields (31).

In particular, the distorted velocity may be defined as follows:

$$u^* - iv^* = e^{\lambda - i\theta} = q^*e^{-i\theta} \tag{32}$$

Then

$$\left.\begin{aligned}\frac{\partial\varphi}{\partial u^*} &= -T\frac{\partial\psi}{\partial v^*}\\ \frac{\partial\varphi}{\partial v^*} &= T\frac{\partial\psi}{\partial u^*}\end{aligned}\right\} \tag{33}$$

Since T is a given function of $u^{*2} + v^{*2} = q^{*2}$, the second-order equations obtained from (33) are

$$\Delta\varphi - 2\,\frac{T'}{T}\left(u^*\,\frac{\partial\varphi}{\partial u^*} + v^*\,\frac{\partial\varphi}{\partial v^*}\right) = 0$$

$$\Delta\psi + 2\,\frac{T'}{T}\left(u^*\,\frac{\partial\psi}{\partial u^*} + v^*\,\frac{\partial\psi}{\partial v^*}\right) = 0$$

where

$$\Delta = (\partial^2/\partial u^{*2}) + (\partial^2/\partial v^{*2}) \quad \text{and} \quad T' = dT/dq^{*2}$$

Distorted Hodograph of a Flow around a Profile

Only the following types of flow will be considered in this paper. The flow covers the domain exterior to a profile P (domain $E(P)$). At infinity the flow approaches a uniform flow in the positive x-direction, so that

$$\lim_{x^2+y^2\to\infty} u = q_\infty > 0, \qquad \lim_{x^2+y^2\to\infty} v = 0 \tag{34}$$

The flow is everywhere subsonic ($M < 1$). The profile P is a streamline of the flow; P is a piecewise analytic curve possessing at most two sharp edges. If there are sharp edges, the Kutta-Joukowski condition is satisfied. There are exactly two stagnation points, both situated on P. A uniform flow $u \equiv$ constant, $v \equiv 0$ is excluded. (For the sake of mathematical discussion it is convenient to admit as "profiles" P curves which intersect themselves in a finite number of points. The exterior $E(P)$ is then a partly multiply covered Riemann surface.)

Incompressible flows considered will be subject to the same restrictions, except that the edges need not be sharp.

The transformation

$$u = u(x,y), \qquad v = v(x,y) \tag{35}$$

takes $E(P)$ into a domain H of the $(u,-v)$-plane; H (the hodograph of the flow) is, in general, multiply covered. The transformation

$$u^* = \frac{q^*}{q} u = q^* \cos \theta, \quad v^* = \frac{q^*}{q} v = q^* \sin \theta \tag{36}$$

takes $E(P)$ into a domain H^* of the $(u^*,-v^*)$-plane. This domain will be denoted as the distorted hodograph of the flow (q^* and $-\theta$ are the polar coordinates in the $(u^*,-v^*)$-plane). (Cf. fig. 3, (a), (b), (c).)

It is known that in the case of an incompressible fluid the mapping of the flow into its hodograph is conformal. It will be shown that the mapping of $E(P)$ into H^* can be considered as conformal if angles and distances in $E(P)$ are measured by means of a certain Riemannian metric generated by the flow.

Transformations Conformal with Respect to a Given Flow

Given a subsonic compressible flow covering a domain D of the (x,y)-plane. Let $ds = (dx, dy)$ be an infinitesimal line element situated at a point x,y of D, M the value of the Mach number at this point, and χ the angle between ds and the streamline passing through x,y. The non-Euclidean length of the line element ds shall be defined as

$$dS = ds\sqrt{1 - M^2 \sin^2 \chi} \tag{37}$$

Let α be the angle between the line element ds and the x-axis. Since $\chi = \pm(\theta - \alpha)$ and $dx = ds \cos \alpha$, $dy = ds \sin \alpha$, it is easily seen that (37) can be written in the form

$$dS^2 = e\, dx^2 + 2f\, dx\, dy + g\, dy^2 \tag{38}$$

where

$$\left.\begin{aligned} e &= 1 - M^2 \sin^2 \theta \\ f &= M^2 \sin \theta \cos \theta \\ g &= 1 - M^2 \cos^2 \theta \end{aligned}\right\} \tag{39}$$

Thus (37) is a Riemann metric.

The non-Euclidean angle A between two line elements $ds = (dx, dy)$, $\delta s = (\delta x, \delta y)$ situated at the same point x,y is defined by

$$\cos A = \frac{e\, dx\, \delta x + f(dx\, \delta y + dy\, \delta x) + g\, dy\, \delta y}{dS\, \delta S}$$

δS being the non-Euclidean length of δs.

A transformation

$$\xi = \xi(x,y), \quad \eta = \eta(x,y) \tag{40}$$

of D into a (simply or multiply covered) domain Δ of the (ξ,η)-plane will be called <u>conformal with respect to the flow</u> if it preserves the sense of rotation and takes each non-Euclidean angle A (in the (x,y)-plane) into the Euclidean angle A. An equivalent definition is the following. The transformation (40) is conformal with respect to the flow if it preserves the sense of rotation and if the ratio

$$\Lambda = \frac{d\sigma}{dS}$$

where dS is the non-Euclidean length of a line element ds in the (x,y)-plane and $d\sigma$, the Euclidean length of its image in the (ξ,η)-plane, depends only upon the position (but not upon the direction) of ds. The symbol Λ is called the <u>local factor of magnification</u>.

If D is mapped conformally with respect to the flow into Δ, and Δ is mapped conformally (in the ordinary sense) into Δ', the resulting mapping of D into Δ' is conformal with respect to flow. Conversely, if D is mapped conformally with respect to the flow into both Δ and Δ', the resulting mapping of Δ into Δ' is conformal in the ordinary sense.

<u>Any transformation (40) of $E(P)$ into the (ξ,η)-plane is conformal with respect to the flow if the potential function φ and the stream function ψ considered as functions of ξ and η satisfy the differential equations</u> (31).

This follows from lemma 1 proved in the appendix by setting $A = 1/\rho$, $B = T$.

If the foregoing result is used and it is noted that φ and ψ satisfy equations (33), the following important theorem is seen to be true:

<u>The mapping of the exterior $E(P)$ of a profile P into the distorted hodograph of a subsonic compressible flow around P is conformal with respect to this flow.</u>

Mapping of a Compressible Flow into a Domain

Exterior to an Arbitrary Profile

The distorted hodograph of a flow around an airfoil is a simply connected Riemann surface bounded by a closed curve (the image of P). By a known theorem of function theory, it is possible to map H^* conformally into an arbitrary simply connected domain. Therefore, it is possible to map $E(P)$ conformally with respect to the flow into the domain $E(\Pi)$ of the (ξ,η)-plane, exterior to a given profile Π. The mapping can be chosen so that the point $\xi + i\eta = \infty$ corresponds to the point $x + iy = \infty$ and that the horizontal direction at infinity is preserved (i.e., at ∞ the direction parallel to the x-axis is taken into the direction parallel to the ξ-axis). Furthermore, by eventually changing the size but not the shape of Π it is possible to obtain a mapping for which the local factor of magnification is equal to 1 at infinity. Since at infinity the metric (37) approaches the metric with constant coefficients

$$dS^2 = dx^2 + (1 - M_\infty^2)dy^2$$

the above conditions mean that

$$\xi^2 + \eta^2 \rightarrow \infty, \quad \frac{\partial \xi}{\partial x} \rightarrow 1, \quad \frac{\partial \xi}{\partial y} \rightarrow 0$$

$$\frac{\partial \eta}{\partial x} \rightarrow 0, \quad \frac{\partial \eta}{\partial y} \rightarrow \sqrt{1 - M_\infty^2} \tag{41}$$

as

$$x^2 + y^2 \rightarrow \infty$$

A transformation satisfying these conditions will be called <u>normalized</u>.

If $E(P)$ is mapped conformally with respect to the flow into $E(\sqcap)$, the resulting correspondence between the points of H^* and those of $E(\sqcap)$ is conformal. Therefore, $u^* - iv^*$ is a one-valued analytic function of $\xi + i\eta$ and can be developed in a Laurent series for sufficiently large values of $|\xi + i\eta|$. The following result will be used later.

Lemma 2: *If the mapping of* $E(P)$ *into* $E(\sqcap)$ *is normalized, then*

$$u^* - iv^* = q^*_\infty - \frac{i\Gamma_1}{2\pi}\,\frac{1}{\xi + i\eta} + \ . \ . \ . \tag{42}$$

where

$$\Gamma_1 = \sqrt{1 - M^2_\infty}\,\frac{q^*_\infty}{q_\infty}\,\Gamma \tag{43}$$

and

$$\Gamma = \int u\,dx + v\,dy \tag{44}$$

is the circulation of the compressible flow.

The proof will be found in the appendix.

Conjugate Flows

Given a compressible flow (in the (x,y)-plane) around the profile P and an incompressible flow (in the (ξ,η)-plane) around a profile $\sqcap$. The complex potential of the incompressible flow will be denoted by $\Omega(\zeta) = \varphi_i + i\psi_i$. Its complex velocity is

$$u_i - iv_i = q_i e^{-i\theta_i} = \frac{d\Omega}{d\zeta}$$

Since it is assumed that at infinity $\theta_i = 0$, Ω can be written in the form

$$\Omega(\zeta) = q_{i,\infty}\,G(\zeta), \qquad G'(\infty) = 1$$

The two flows will be called *conjugate* (*modulo* n), if there exists a real number n,

$$0 < n < 2$$

such that the transformation

$$u_i - iv_i = (u^* - iv^*)^n \tag{45}$$

takes the distorted hodograph H^* of the compressible flow into the hodograph H_i of the incompressible flow. The connection between conjugate flows is shown in figure 3.

The mapping (45) defines a mapping of $E(P)$ into $E(\Gamma)$. This mapping is conformal with respect to the flow around P. For so is the mapping of $E(P)$ into H^*; and the mapping of H^* into H_i (given by (45) as well as the mapping of H_i into $E(\Gamma)$ are conformal in the ordinary sense. The mapping of $E(P)$ into $E(\Gamma)$ preserves the point infinity and the horizontal direction at infinity (for at infinity both flows are horizontal). On P (on Γ) $\theta(\theta_i)$ is the slope of the profile. According to (45) the slopes at corresponding points are connected by the relation

$$\theta_i = n\theta \tag{46}$$

(The slope is defined as the angle between the tangent to the profile and the positive x-axis; the tangent pointing in the direction of the flow.)

Conversely, if it is possible to map $E(P)$ into the domain $E(\Gamma)$ in the (ξ,η)-plane exterior to a profile Γ, by a transformation which is conformal with respect to the compressible flow around P, which preserves the point ∞ and the horizontal direction at infinity, and which changes the slope of P according to (46), then the flow around P is conjugate (modulo n) to an incompressible flow around Γ which has stagnation points at the points into which the stagnation points at P are taken and the direction at infinity of which is horizontal (provided such a flow exists).

For, let $G(\zeta)$ be the complex potential of such a flow, $G'(\infty) = 1$. Set

$$u_i - iv_i = q^{*n}_{\infty} G'(\zeta)$$

The mapping of $E(\Gamma)$ into H^* is conformal - that is, $u^* - iv^*$ is a one-valued analytic function defined in $E(\Gamma)$.

Therefore, $\theta = -\text{Im} \log(u^* - iv^*)$ is harmonic in $E(\Gamma)$; $n\theta$ coincides on Γ with the harmonic function $\theta_i = -\text{Im} \log(u_i - iv_i)$. Therefore, (46) holds throughout $E(\Gamma)$. Since $n \log q^*$ is conjugate to $n\theta$ and $\log q_i = \log |u_i - iv_i|$ to θ_i

$$\log q_i = n \log q^* + \text{constant}$$

The above constant must vanish, for at infinity

$$q_\infty^{*n} = q_{i,\infty}$$

Therefore

$$q^{*n} = q_i \tag{47}$$

and by (46) and (47), (45) holds.

If an incompressible flow is conjugate to a subsonic compressible flow, then

$$\frac{q_{i,max}}{q_{i,\infty}} < K \tag{48}$$

where $q_{i,max}$ is the maximum speed and K depends upon n and the stream Mach number M_∞ of the compressible flow. For q^*_{max} must be less than 1 (cf. (15)) and therefore, by (47)

$$\frac{q_{i,max}}{q_{i,\infty}} = \left(\frac{q^*_{max}}{q^*_\infty}\right)^n < q^{*-n}_\infty \tag{49}$$

Note that q^*_∞ is a function of q_∞ and therefore also of M_∞.

If the incompressible flow with the complex potential $\Omega(\zeta)$ is conjugate to a given compressible flow, so is the flow with the complex potential

$$\frac{1}{A}\,\Omega(A\zeta), \quad A > 0$$

for it has the same hodograph. Thus the conjugate profile Π can be chosen so that the mapping of $E(P)$ into $E(\Pi)$ is normalized.

If the incompressible flow around Π conjugate to a compressible flow around P is known, then the velocity distribution (and therefore the pressure distribution) around P can be immediately computed. For (46) gives the correspondence between the points of Π and P, and the speeds at corresponding points are given by (47). However, the present method has not been developed sufficiently to permit a solution of the direct problem: to find the incompressible flow conjugate to a compressible flow around a given profile P. The following sections contain the solution of the inverse problem: to find a compressible flow around a closed profile conjugate to a given incompressible flow, and the discussion of the existence of conjugate flows, which is by no means self-evident.

II.- CONSTRUCTION OF SUBSONIC FLOWS AROUND A PROFILE UNDER THE ASSUMPTION OF THE LINEARIZED EQUATION OF STATE

Simplifications Resulting from the Assumption $\gamma = -1$

Throughout this chapter the pressure-density relation is assumed to have the linearized form - that is, γ is set equal to -1. Under this assumption it can be shown that under very general conditions each compressible flow possesses a conjugate incompressible flow and vice versa. The inverse problem can be solved completely and in a closed form.

The assumption $\gamma = -1$ implies the following simplifications.

1. The differential equation of the potential φ in the physical plane takes the form

$$(1 + v^2) \frac{\partial^2 \varphi}{\partial x^2} - 2uv \frac{\partial^2 \varphi}{\partial x \partial y} + (1 + u^2) \frac{\partial^2 \varphi}{\partial y^2} = 0$$

$$u = \frac{\partial \varphi}{\partial x}; \quad v = \frac{\partial \varphi}{\partial y} \tag{50}$$

(Cf. (32) and (10).) Thus the surface

$$z = \varphi(x,y) \tag{51}$$

(potential surface) is a <u>minimal surface</u> (surface of vanishing mean curvature).

2. The non-Euclidean length dS of a line element ds in the (x,y)-plane (cf. sec. Transformations Conformal with Respect to a Given Flow) becomes equal to

$$dS = dl\sqrt{1 - M^2}$$

where dl is the Euclidean length of the line element on the potential surfaces the projection of which is ds.

For, if the angle between the direction of the velocity vector and ds is denoted by χ, it follows from (37) that

$$\begin{aligned} dl^2 &= (1 + u^2)\,dx^2 + 2uv\,dx\,dy + (1 + v^2)\,dy^2 \\ &= ds^2 + (u\,dx + v\,dy)^2 \\ &= ds^2 + ds^2\,q^2\cos^2\chi \\ &= \frac{ds^2\,(1 - M^2\sin^2\chi)}{1 - M^2} \\ &= \frac{dS^2}{1 - M^2} \end{aligned}$$

3. The term $\varphi + i\psi$ is an analytic function of $u^* - iv^*$ and of the complex variable in any plane into which $E(P)$ is mapped conformally with respect to the flow. (Cf. (30), (33).)

Existence of Conjugate Flows

It will be shown presently that to any compressible flow around a profile P in the z-plane (obeying the linearized equation of state) there exists a conjugate flow of an incompressible fluid around a profile Γ in the ζ-plane, provided either of the following two conditions is satisfied.

(a) The compressible flow is circulation-free. In this case the modulus n can be chosen at random. In particular, it is convenient to set $n = 1$.

(b) The compressible flow possesses a circulation and the Mach number at infinity is restricted by

$$M_\infty < \sqrt{3/4} = 0.866 \ldots \tag{52}$$

In this case the modulus n is given by

$$n = \frac{1}{\sqrt{1 - M_\infty^2}} \tag{53}$$

(Cf. fig. 4.)

For the proof, map $E(P)$ into the exterior of a circle $X^2 + Y^2 = R^2$ in the Z-plane ($Z = X + iY$, R being a conveniently chosen constant) by a normalized transformation which is conformal with respect to the given flow. Then the correspondence between the distorted hodograph H^* of the compressible flow and the domain $|Z| > R$ is conformal - that is, $u^* - iv^*$ is an analytic function of Z. By lemma 2, this function has the form

$$u^* - iv^* = q^*_\infty - \frac{i\Gamma_1}{2\pi}\frac{1}{Z} + \ldots \tag{54}$$

Γ_1 being given by (43). Furthermore, $\varphi + i\psi$ is an analytic function of $u^* - iv^*$ and therefore also of Z, $|Z| > R$; $\psi = 0$ on $|Z| = R$ for $\psi = 0$ on P. If Z goes once around the circle $|Z| = R$, $\varphi + i\psi$ increases by Γ. Next,

$$\lim_{Z \to \infty} \frac{\partial\varphi}{\partial X} = \lim_{Z \to \infty} \left\{\frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial X} + \frac{\partial\varphi}{\partial y}\frac{\partial y}{\partial X}\right\} = q_\infty$$

$$\lim_{Z \to \infty} \frac{\partial\varphi}{\partial Y} = \lim_{Z \to \infty} \left\{\frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial Y} + \frac{\partial\varphi}{\partial y}\frac{\partial y}{\partial Y}\right\} = 0$$

for the mapping is assumed to be normalized. It follows that

$$\varphi + i\psi = q_\infty\left(Z + \frac{R^2}{Z}\right) - \frac{i\Gamma}{2\pi}\log Z + \text{constant} = \Omega_1(Z)$$

Thus $\varphi + i\psi$ is in the Z-plane the complex potential of a flow around the circle $|Z| = R$. This flow necessarily possesses two stagnation points on the circle $|Z| = R$; namely, the images of the stagnation points on P. For at these points the lines ψ = constant intersect the circle. It follows that

$$\frac{d(\varphi + i\psi)}{dZ} = q_\infty\left(1 - \frac{R^2}{Z^2}\right) - \frac{i\Gamma}{2\pi}\frac{1}{Z} \tag{55}$$

vanishes at the same points $(Z = S_1,\ Z = S_2,\ S_1 = -\overline{S}_2)$ as does $u^* - iv^*$.

Assume that there exists an incompressible flow around a profile $\sqcap$ in the ζ-plane which is conjugate (modulo n) to the compressible flow around P. Let

$$Z = Z(\zeta)$$

map $E(\sqcap)$ conformally into $|Z| > R$ taking $\zeta = \infty$ into $Z = \infty$. Without loss of generality it may be assumed that

$$Z'(\infty) = 1$$

Then the complex potential $\Omega(\zeta) = \Omega_i + i\psi_i$ of the incompressible flow must in the Z-plane be of the form

$$\Omega = \frac{q_{i,\infty}}{q_\infty}(\varphi + i\psi)$$

The complex velocity of the conjugate flow, $u_i - iv_i$, is given by

$$u_i - iv_i = \frac{d\Omega}{d\zeta}$$

On the other hand,

$$u_i - iv_i = (u^* - iv^*)^n$$

Therefore

$$\frac{d\zeta}{dZ} = \frac{d\Omega}{dZ}\bigg/\frac{d\Omega}{d\zeta} = \frac{q_\infty^{*n}}{q_\infty}\,\frac{d(\varphi + i\psi)}{dZ}\bigg/(u^* - iv^*)^n$$

so that

$$\zeta(Z) = \int q_\infty^{*n}\,\frac{1 - R^2/Z^2 - i\Gamma/2\pi\, q_\infty Z}{(u^* - iv^*)^n}\,dZ \tag{56}$$

The numerator of the integrand in the above formula possesses simple zeros at the two stagnation points. It can be easily shown that at the two stagnation points (in the Z-plane) $u^* - iv^*$ vanishes of an order not higher than 1. For at these points $\text{Im}\log(u^* - iv^*) = -\theta$ possesses jumps of magnitude $\epsilon_1\pi$, $\epsilon_1\pi$ and $\epsilon_2\pi$ being the angles at the stagnation points on P. Therefore $\text{Re}\log(u^* - iv^*) = \log q^*$ behaves at these points as $\epsilon_i\log|Z - S_i|$. And it has been assumed that P possesses only sharp edges if any, so that $0 < \epsilon_i \leq 1$. It follows that at S_1 and S_2 the integrand becomes infinite of an order less than 1, provided $0 < n < 2$. At all other points Z, $|Z| \geq R$, the integrand is different from both 0 and ∞. Hence the integration can be performed also along the circle $|Z| = R$.

Now, since Γ was assumed to be a closed curve, the integral (56) must vanish if the integration is performed along the closed circle $|Z| = R$. By (54)

$$\frac{1}{(u^* - iv^*)^n} = \frac{1}{q_\infty^{*n}}\left(1 + \frac{n}{q_\infty^*}\,\frac{i\Gamma_1}{2\pi}\,\frac{1}{Z} + \ldots\right)$$

so that the integrand in (56) equals

$$1 + \left(\frac{n\Gamma_1}{q_\infty^*} - \frac{\Gamma}{q_\infty}\right)\frac{i}{2\pi}\,\frac{1}{Z} + \ldots$$

In order that the integral taken along a closed curve should vanish it is necessary and sufficient that

$$\frac{n\Gamma_1}{q_\infty^*} - \frac{\Gamma}{q_\infty} = 0$$

or, by virtue of (43), that

$$\left(n\sqrt{1 - M_\infty^2} - 1\right)\Gamma = 0 \tag{57}$$

This condition will be satisfied if either $\Gamma = 0$ or n is determined according to (53). In this last case the condition $n < 2$ yields the bound (52) for M_∞.

Conversely, if (57) holds and $n < 2$, the function (56) maps $|Z| > R$ conformally into the exterior of a closed profile $\sqcap$ in the ζ-plane. (Note that the derivative of this function does not vanish.) This mapping satisfies the conditions

$$\zeta(\infty) = \infty, \quad \zeta'(\zeta) = 1$$

The resulting mapping of $E(P)$ into $E(\sqcap)$ is conformal with respect to the flow around P. Obviously,

$$\frac{q_\infty^{*n}}{q_\infty} \Omega_1\left[Z(\zeta)\right]$$

is the complex potential of an incompressible flow around $\sqcap$. The complex velocity of this flow is equal to $(u^* - iv^*)^n$. Thus the flow is conjugate to the compressible flow around P and the assertions formulated in the beginning of this section are proved.

Properties of the Conjugate Flow

From the construction of the conjugate flow given in the preceding section, it follows that the circulation of the conjugate incompressible flow is equal to

$$\frac{q_\infty^{*n}}{q_\infty}\Gamma = \frac{M_\infty^{n-1}\sqrt{1 - M_\infty^2}}{\left(1 + \sqrt{1 - M_\infty^2}\right)^n}\Gamma$$

(Γ being the circulation of the compressible flow), provided $\sqcap$ has been chosen so that the mapping of $E(P)$ into $E(\sqcap)$ is normalized.

For the conjugate incompressible flow the ratio $q_{i,max}/q_{i,\infty}$ cannot exceed $K = q_{\infty}^{*-n}$. (Cf. end of pt. I.) For $\gamma = -1$,

$$\left.\begin{aligned} K &= \frac{1 + \sqrt{1 - M_{\infty}^2}}{M_{\infty}} \quad \text{for} \quad n = 1 \\ K &= \left\{\frac{1 + \sqrt{1 - M_{\infty}^2}}{M_{\infty}}\right\}^{1/\sqrt{1-M_{\infty}^2}} \quad \text{for} \quad n = 1/\sqrt{1 - M_{\infty}^2} \end{aligned}\right\} \tag{58}$$

(The values of K are plotted in fig. 5.)

It should be noted that the profile Γ constructed in the preceding section is necessarily closed but need not be simple - that is, Γ might intersect itself, in which case $E(\Gamma)$ would be partly multiply covered. For this reason such physically impossible flows were included in the discussion. It is easily seen that Γ will always be simple if P is convex.

Solution of the Inverse Problem

Suppose that an incompressible flow around a profile Γ in the ζ-plane is given and it is known that this flow is conjugate (modulo n) to a compressible flow around a profile P in the z-plane. This section contains the derivation of the formulas which permit finding P and the compressible flow around P. It will turn out that these formulas always yield a compressible flow around a closed profile, even if Γ and the flow around Γ are chosen at random.

Let the complex potential of the incompressible flow be given in the form

$$\Omega(\zeta) = q_{i,\infty}\, G(\zeta), \quad G'(\infty) = 1$$

Since $\varphi + i\psi$ (complex potential of the compressible flow) considered as a function of ζ is analytic and real on Γ and since $d(\varphi + i\psi)/d\zeta$ vanishes at the stagnation points of the incompressible flow,

$$\varphi + i\psi = CG(\zeta)$$

C being a positive constant. Without loss of generality it may be assumed that the mapping of $E(P)$ into $E(\Gamma)$ is normalized. Then

$$C = q_\infty$$

Now, let ds be a line element on P and $d\sigma$ the corresponding line element on Γ. Then

$$\frac{d\varphi}{d\sigma} = C \frac{dG}{d\sigma} = \frac{q_\infty}{q_{i,\infty}} q_i$$

On the other hand, on P

$$\frac{d\varphi}{ds} = q$$

Since the increase of φ on ds is equal to the increase of φ on $d\sigma$, and since $q_i = q^{*n}$,

$$\frac{ds}{d\sigma} = \frac{q_\infty}{q_{i,\infty}} \frac{q_i}{q} = \frac{q_\infty}{q_\infty^{*n}} \frac{q^{*n}}{q}$$

By (18)

$$\frac{q^*}{q} = \frac{1}{2} (1 - q^{*2})$$

so that

$$\frac{ds}{d\sigma} = \frac{q^{*n-1} - q^{*n+1}}{q_\infty^{*n-1} - q_\infty^{*n+1}}$$

$$= \frac{q_i^{1-1/n} - q_i^{1+1/n}}{q_\infty^{*n-1} - q_\infty^{*n+1}} \tag{59}$$

Since

$$q_i = q_{i,\infty}|G'(\zeta)| = q^{*n}|G'(\zeta)|$$

(59) can be written as

$$\frac{ds}{d\sigma} = \frac{1}{1 - \mu^2}\left\{|G'(\zeta)|^{1-1/n} - \mu^2 |G'(\zeta)|^{1+1/n}\right\} \tag{60}$$

where $\mu = q^*_\infty$ – that is,

$$\mu = \frac{M_\infty}{1 + \sqrt{1 - M_\infty^2}} \tag{61}$$

Next, let dz and $d\zeta$ be the complex line elements on P and $\sqcap$, respectively:

$$dz = ds\ e^{i\theta} = ds\ e^{i\theta_1/n}$$

$$d\zeta = d\sigma\ e^{i\theta_1}$$

Then

$$dz = \frac{ds}{d\sigma}\, d\sigma\ e^{i\theta_1/n} \tag{62}$$

Since

$$|G'(\zeta)| = G'(\zeta)\ e^{i\theta_1} = \overline{G'(\zeta)}\ e^{-i\theta_1}$$

there is obtained by (60) and (62)

$$dz = \frac{1}{1 - \mu^2}\left\{G'(\zeta)^{1-1/n}\, d\zeta - \mu^2\, \overline{G'(\zeta)^{1+1/n}\, d\zeta}\right\}$$

Integration yields the following representation of the profile P:

$$z = \text{constant}\left\{\int G'(\zeta)^{1-1/n}\, d\zeta - \mu^2\, \overline{\int G'(\zeta)^{1+1/n}\, d\zeta}\right\} \tag{63}$$

the integration being performed along $\sqcap$. (The value of the constant factor affects only the size of P.)

For a circulation-free flow and $n = 1$ this formula simplifies to

$$z = \text{constant}\left\{\zeta - \mu^2\, \overline{\int G'^2\, d\zeta}\right\} \tag{64}$$

(64) and (61) are exactly the formulas given in Tsien's paper (reference 14).

Parametric Representation of Subsonic Compressible Flows

Formula (63) has been derived under the assumption that the integration is being performed along Γ and that the existence of a conjugate compressible flow is known beforehand. Both conditions are unessential. For the following general result holds:

Let $G(\zeta)$ be the complex potential of an incompressible flow around a profile Γ in the ζ-plane,

$$G'(\infty) = 1 \tag{65}$$

Let M_∞ be a real number such that

$$0 < M_\infty < 1 \quad \text{if } G \text{ is one-valued,}$$

$$0 < M_\infty < \sqrt{3/4} \quad \text{if } G \text{ is multi-valued.}$$

Set

$$\mu = \frac{M_\infty}{1 + \sqrt{1 - M_\infty^2}} \tag{66}$$

$$n = \begin{cases} 1 + \delta, \quad 0 \leq \delta < 1, & \text{if } G \text{ is one-valued} \\ 1/\sqrt{1 - M_\infty^2} & \text{if } G \text{ is multi-valued} \end{cases} \tag{67}$$

$$K = \mu^{-n} \tag{68}$$

If for $\zeta \in E(\Gamma)$,

$$|G'(\zeta)| < K \tag{69}$$

then the function

$$z = C\left\{\int G'(\zeta)^{1-1/n}\, d\zeta - \mu^2 \int \overline{G'(\zeta)^{1+1/n}\, d\zeta}\right\} \tag{70}$$

$C > 0$, *maps* $E(\Gamma)$ *in a one-to-one manner into the exterior* $E(P)$ *of a closed profile* P *in the z-plane taking* $\zeta = \infty$ *into* $z \rightarrow \infty$, *and*

$$\varphi = 2C\mu \operatorname{Re} G(\zeta) \tag{71}$$

considered as a function of x *and* y *is the potential of a subsonic compressible flow around* P (obeying the linearized equation of state) *of stream Mach number* M_∞.

The two flows are conjugate (modulo n). The mapping of $E(P)$ *into* $E(\Gamma)$ *is normalized by choosing* $C = 1/(1 - \mu^2)$.

The proof of the mapping properties of the function (70) will be found in the appendix under C. Since the mapping of $E(\Gamma)$ into $E(P)$ is one-to-one, φ may be considered as a function of (x,y). Equations (70) and (71) may be rewritten in the form

$$x = \operatorname{Re} f(\zeta), \quad y = \operatorname{Re} g(\zeta), \quad \varphi = \operatorname{Re} h(\zeta) \tag{72}$$

where the analytic functions f, g, and h are given by

$$f(\zeta) = C\int\left(G'^{1-1/n} - \mu^2 G'^{1+1/n}\right) d\zeta$$

$$g(\zeta) = -iC\int\left(G'^{1-1/n} + \mu^2 G'^{1+1/n}\right) d\zeta$$

$$h(\zeta) = 2C\mu G$$

Since

$$f'^2 + g'^2 + h'^2 = 0$$

(72) is the well-known Weierstrass parameter representation of a minimal surface. Hence, $\varphi(x,y)$ satisfies equation (50) and therefore is a potential of a compressible flow.

It is shown in the appendix under C that as $\zeta \rightarrow \infty$

$$\frac{\partial x}{\partial \xi} \rightarrow C(1 - \mu^2), \quad \frac{\partial x}{\partial \eta} \rightarrow 0, \quad \frac{\partial y}{\partial \xi} \rightarrow 0, \quad \frac{\partial y}{\partial \eta} \rightarrow C(1 + \mu^2)$$

(Cf. (C3).) By (65) and (71) this implies that as $z \longrightarrow \infty$

$$\frac{\partial\varphi}{\partial x} \longrightarrow \frac{2\mu}{1 - \mu^2}, \quad \frac{\partial\varphi}{\partial y} \longrightarrow 0$$

Thus, at infinity the compressible flow is parallel to the x-axis and

$$q_\infty = \frac{2\mu}{1 - \mu^2} = \frac{M_\infty}{\sqrt{1 - M_\infty^2}}$$

so that M_∞ is actually the stream Mach number.

It is also shown in the appendix under C that the direction normal to Π is taken into the direction normal to P. Since the normal derivative of φ on Π vanishes, so does the normal derivative of φ on P. Thus P is a streamline of the compressible flow.

The fact that the two flows are conjugate follows simply by comparing the velocities at corresponding points of P and Π. The details of this computation may be omitted.

From the preceding section it follows that the parametric representation (70) and (71) yields <u>all</u> flows satisfying the conditions stated in the beginning of part II. (Note that neither Π nor P are necessarily simple curves.)

Construction of a Compressible Flow around a Profile Similar to a Given Profile

Suppose a profile P_1 and a point S on this profile are given and it is desired to find a subsonic compressible flow around a profile P similar to P_1, possessing a prescribed stream Mach number $M_\infty (M_\infty < \sqrt{3/4})$ and a stagnation point near the point S. (Since S is determined by the angle of attack the last requirement determines approximately the position of P with respect to the undisturbed flow.) This problem can be solved as follows.

For the sake of definiteness it will be assumed that the profile P_1 has one sharp (trailing) edge. It may be assumed that the function mapping P_1 conformally into a circle is known. In fact, this function can be easily computed. (See reference 13.) The first step consists in forming an incompressible flow around P_1 which possesses stagnation points at the trailing edge and at S. The direction of this flow at infinity is taken as the ξ-direction in the plane of P_1 (ζ-plane). Now, if $E(P_1)$ is mapped into $|Z| > 1$ by an analytic function

$$Z = F(\zeta) \tag{73}$$

which satisfies the conditions

$$Z(\infty) = \infty, \qquad Z'(\infty) > 0$$

then the sharp trailing edge and the point S are taken into the points $e^{-i\alpha}$, $-e^{i\alpha}$, respectively, α being real (cf. fig. 6).

From M_∞ is determined the modulus n by (53). Now, let C_1 and C_2 be two circles passing through the points $e^{-i\alpha}$ and $-e^{i\alpha}$ which intersect at the angle $n\pi$. The function

$$\zeta = F_1(Z)$$

inverse to (73) maps the infinite domain bounded by an arc of C_1 and an arc of C_2 into the exterior of some closed profile Π. Profile Π possesses two singular points: the trailing edge which coincides with the trailing edge of P_1 and the point S. The angles there are $n\beta$ and $n\pi$, β being the angle at the trailing edge of P_1.

Since a domain bounded by two circular arcs can easily be mapped into the exterior of a circle, it is easy to compute incompressible flows around Π.

From M_∞ is determined q_∞ (by (11)) and q^*_∞ (by (18)). Let $G(\zeta)$ be the complex potential of an incompressible flow around Π which satisfies the Kutta-Joukowski condition at the trailing edge and the condition

$$G'(\infty) = 1$$

It is easily seen that this flow possesses a stagnation point at S. The same is true for the flow with the complex potential

$$q_{i,\infty}\, G(\zeta), \quad q_{i,\infty} = q_\infty^{*n}$$

Now a compressible flow around a profile P is constructed which is conjugate (modulo n) to the above incompressible flow; P is given by (70). The velocity potential is given by (71). The distorted speed q^* of the compressible flow at a point z of P is equal to

$$q^* = q_i^{1/n}$$

q_i being the speed of the incompressible flow at the corresponding point of $\sqcap$. From q^*, q is determined by (18).

Since the angle between the x-axis and the tangent to P at a point z is equal to $1/n$ times the angle between the ξ-axis and the tangent to $\sqcap$ at the corresponding point ζ, it is seen that the profile P will possess only one sharp edge, the angle there being β.

The profile distortion (i.e., the difference between P_1 and P) is due (1) to the difference between P_1 and $\sqcap$ and (2) to the difference between $\sqcap$ and P. This distortion will be small if M_∞ is not too large. For then μ^2 is small and n close to 1. (See table II; in fig. 4, n and μ^2 are plotted as functions of M_∞.) Therefore, the circles C_1, C_2 are close to the unit circle and $\sqcap$ close to P_1. Secondly, $1-1/n$ will be small and therefore the first term in (70) will be close to ζ while the second term will be small as compared to the first.

Of course, it is possible to construct $\sqcap$ in many other ways. If the flow around P_1 which has a stagnation point at S is circulation-free (i.e., if $\alpha = 0$), it is possible to set $n = 1$. Then P_1 coincides with $\sqcap$.

Alternative Formulas

It is useful to write the formulas transforming an incompressible flow into a compressible flow in a different form. The (dimensionless) speed at infinity q_∞ will be used as the parameter characterizing the flow.

From formulas (52), (61), and (11) it follows that

$$n = \sqrt{1 + q_\infty^2} \tag{74}$$

$$\mu^2 = \frac{n - 1}{n + 1} \tag{75}$$

The function mapping the profile Π into the profile P takes the form

$$z = \frac{n + 1}{2} \int G'(\zeta)^{1-1/n}\, d\zeta - \frac{n + 1}{2} \overline{\int G'(\zeta)^{1+1/n}\, d\zeta} \tag{76}$$

The potential at the compressible flow is given by

$$\varphi = \sqrt{n^2 - 1}\; G \tag{77}$$

(The arbitrary constant appearing in (70) and (71) has been chosen as $1/(1 - \mu^2) = (n + 1)/2$.) Finally, the speed q of the compressible flow around P is given by

$$q = \frac{2\left(\frac{n - 1}{n + 1}\right)^{\frac{1}{2}} \left|G'(\zeta)\right|^{1/n}}{1 - \left(\frac{n - 1}{n + 1}\right) \left|G'(\zeta)\right|^{2/n}} \tag{78}$$

This follows immediately from formulas (47) and (18), by noting that in this case

$$q_i = u^n \, |G'|$$

Tsien's simpler formulas which are valid for the case of a circulation-free flow may be rewritten similarly.

$$\left.\begin{aligned} z &= \frac{n+1}{2}\zeta - \frac{n-2}{2}\overline{\int G'(\zeta)^2\, d\zeta} \\ q &= \frac{2\left(\dfrac{n-1}{n+1}\right)^{\frac{1}{2}} \left|G'(\zeta)\right|}{1 - \left(\dfrac{n-1}{n+1}\right)\left|G'(\zeta)\right|^2} \end{aligned}\right\} \tag{79}$$

ϑ is given by the same formula as before (formula (77)).

III. CONSTRUCTION OF SUBSONIC FLOWS AROUND A PROFILE UNDER THE ASSUMPTION OF THE ACTUAL EQUATION OF STATE

Existence of Conjugate Flows

In this chapter it is assumed that the pressure-density relation is not of the linearized form but a general one, say the polytropic relation ($\gamma > 1$). It will be shown that the construction of a compressible flow for a given conjugate incompressible flow reduces to the solution of a boundary value problem for a linear partial differential equation in the physical plane of the conjugate flow.

The discussion of the existence of a conjugate flow for a given compressible flow around P can be carried out in the same way as previously. The essential difference consists in the fact that so far it has been impossible to characterize completely all compressible flows possessing conjugate incompressible flows and to determine a priori, the modulus n.

Consider a compressible flow around a profile P in the z-plane. It has been shown that it is possible to map $E(\Pi)$ into the exterior of a circle $|Z| = R$ by a transformation which is conformal with respect to the flow and normalized at infinity. The two stagnation points of the flow are taken into two points

$$Z_1 = Re^{-i\alpha}; \quad Z_2 = Re^{-i(\alpha+\delta)}$$

The flow will be said to satisfy <u>condition</u> <u>A</u> if

$$\delta = \pi - 2\alpha$$

that is, if

$$Z_2 = -\overline{Z}_1$$

Assume that condition A is satisfied, and set

$$\Gamma_2 = -4\pi\, q_\infty R \sin\alpha \tag{80}$$

If

$$0 < \Gamma_2/\Gamma < 2\sqrt{1 - M_\infty^2} \tag{81}$$

(Γ being the circulation of the compressible flow), the flow will be said to satisfy <u>condition</u> <u>B</u>. If $\Gamma = 0$, condition B implies that $\Gamma_2 = 0$.

In the case $\gamma = -1$, $\varphi + i\psi$ is an analytic function of Z; condition A is always satisfied and Γ_2 is equal to Γ. Therefore condition B is satisfied for

$$\Gamma = 0 \quad \text{and} \quad \text{for} \quad \Gamma \neq 0, \quad M_\infty < \sqrt{3/4}$$

It is <u>conjectured</u> (but has not been proved) that A is always satisfied and B is satisfied if M_∞ does not exceed some limiting value.

Condition A is certainly satisfied (by reasons of symmetry) if either of the two geometrical conditions holds:

(i) P is symmetric with respect to the x-axis and the flow is circulation-free.

(ii) Both P and the flow around P are symmetric with respect to the y-axis. (This type of flow includes flows with circulation.)

Flows of type (i) obviously satisfy condition B. In general, it may be assumed that B will be satisfied in many cases. For the flow satisfying the linearized equation of state is a rather good approximation to the flow satisfying the actual equation of state and therefore Γ_2 should not differ too much from Γ.

<u>If a compressible flow satisfies both conditions</u> A <u>and</u> B, <u>then it is conjugate (modulo n) to an incompressible flow around a profile</u> $\sqcap$ <u>in the ζ-plane. If</u> $\Gamma = 0$, n <u>is arbitrary and may be chosen as</u> 1. <u>If</u> $\Gamma \neq 0$, n <u>is given by</u>

$$n = \frac{\Gamma_2}{\Gamma} \frac{1}{\sqrt{1 - M_\infty^2}} \tag{82}$$

The proof is almost exactly the same as the one given for the special case $\gamma = -1$ and it will suffice to sketch the argument.

Let Ω_1 be the complex potential of a flow around $|Z| = R$ which has stagnation points at Z_1 and Z_2 and the velocity q_∞ at infinity. Then Ω_1 has the form

$$\Omega_1 = q_\infty \left(Z + \frac{R^2}{Z}\right) - \frac{i\Gamma_2}{2\pi} \log Z + \text{constant}$$

Γ_2 being given by (80). As before, $u^* - iv^*$ is an analytic function of Z and has (according to Lemma 2) the form (42). If there exists a conjugate (modulo n) incompressible flow in the ζ-plane (around a profile $\sqcap$), its complex potential Ω considered as a function of Z must have the form

$$\Omega = \lambda\Omega_1 \quad (\lambda \text{ a real constant})$$

As before, the function which maps $|Z| > R$ into $E(\sqcap)$ will be given by

$$\zeta(Z) = \lambda \int^Z \frac{d\Omega_1}{dZ} (u^* - iv^*)^{-n} \, dZ \tag{83}$$

and the requirement that $\sqcap$ be a closed profile leads to the condition

$$\frac{n\Gamma_1}{q^*_\infty} = \frac{\Gamma_2}{q_\infty}$$

or

$$n\Gamma\sqrt{1 - M_\infty^2} = \Gamma_a \tag{84}$$

which is satisfied if either

$$\Gamma = \Gamma_a = 0$$

or n is determined by (82).

Conversely, if (84) holds, formula (83) yields a closed profile $\sqcap$ (note that by (81), $0 < n < 2$) and the incompressible flow around $\sqcap$ which has the velocity $q_{1,\infty} = q_\infty^{*n}$ at infinity and stagnation points at $\zeta(Z_1)$ and $\zeta(Z_2)$, is conjugate (modulo n) to the given compressible flow.

It can be easily seen that the conditions A, B are not only sufficient but also necessary for the existence of a conjugate flow.

Solution of the Inverse Problem

Suppose that an incompressible flow around the profile $\sqcap$ in the ζ-plane is conjugate (modulo n) to a compressible flow around the profile P in the z-plane. It has been shown that φ and ψ (potential and stream function of the compressible flow) satisfy the equations (33) in the distorted hodograph plane. Since the correspondence between the distorted hodograph H^* and $E(\sqcap)$ is conformal, it follows that φ and ψ, considered as functions of ξ and η, satisfy the equations

$$\left.\begin{aligned} \frac{\partial\varphi}{\partial\xi} &= T\frac{\partial\psi}{\partial\eta} \\ \frac{\partial\varphi}{\partial\eta} &= -T\frac{\partial\psi}{\partial\xi} \end{aligned}\right\} \tag{85}$$

and the second-order equations

$$\frac{\partial}{\partial\xi}\left\{\frac{1}{T}\frac{\partial\varphi}{\partial\xi}\right\} + \frac{\partial}{\partial\eta}\left\{\frac{1}{T}\frac{\partial\varphi}{\partial\eta}\right\} = 0 \tag{86}$$

$$\frac{\partial}{\partial\xi}\left\{T\frac{\partial\psi}{\partial\xi}\right\} + \frac{\partial}{\partial\eta}\left\{T\frac{\partial\psi}{\partial\eta}\right\} = 0 \tag{87}$$

If the incompressible flow around $\sqcap$ is known, T is a known function of ξ and η. For T is a known function of q (see (27)) and therefore also of q^* (see (14). And at corresponding points of $E(P)$ and $E(\sqcap)$, q^* and q_i (the speed of the incompressible flow) are connected by the relation

$$q_i = q^{*n} \tag{88}$$

Therefore the equations (85) to (87) are <u>linear</u>.

The boundary conditions for φ and ψ are

$$\psi = \text{constant}, \quad \frac{\partial\varphi}{\partial\nu} = 0 \tag{89}$$

$\frac{\partial}{\partial\nu}$ indicating differentiation in the direction normal to $\sqcap$. At infinity φ and ψ must satisfy the conditions

$$\frac{\partial\varphi}{\partial\xi} \longrightarrow C_1 > 0, \quad \frac{\partial\varphi}{\partial\eta} \longrightarrow 0 \tag{90}$$

and

$$\frac{\partial\psi}{\partial\xi} \longrightarrow 0, \quad \frac{\partial\psi}{\partial\eta} \longrightarrow C_2 > 0 \tag{91}$$

where C_1 and C_2 are positive constants. This can be easily verified by noting that the mapping of $E(P)$ into $E(\sqcap)$ preserves the horizontal direction at infinity. The numerical values of these constants are of no consequence, since both the differential equations and the boundary conditions are linear and homogeneous.

Function ψ must always be one-valued; φ is one-valued only if $\Gamma = 0$. Moreover, φ and ψ must satisfy the conditions:

$$\text{grad}^2 \varphi < \infty, \quad \text{grad}^2 \psi < \infty \tag{92}$$

Thus, it is seen that the equations (85) can be interpreted physically as the equations of motion of a compressible fluid of variable density $\tilde{\rho}$ which is a given function of space:

$$\tilde{\rho}\,(\xi,\eta) = \frac{\rho}{\sqrt{1 - M^2}} \tag{93}$$

$\sqcap$ is a streamline of this flow and at infinity the flow has the positive ξ-direction.

Assume that φ is known as a function of ξ and η and set

$$\tilde{q} = \sqrt{\left(\frac{\partial\varphi}{\partial\xi}\right)^2 + \left(\frac{\partial\varphi}{\partial\eta}\right)^2} \tag{94}$$

($\tilde{q}$ is the speed of the fictitious compressible flow of density $\tilde{\rho}$). Let ds be a line element on P and $d\sigma$ the corresponding line element on $\sqcap$. Then

$$\frac{d\varphi}{ds} = q$$

$$\frac{d\varphi}{d\sigma} = \tilde{q}$$

so that

$$\frac{ds}{d\sigma} = \frac{\tilde{q}}{q}$$

Since the angle between ds and the x-axis is θ and that between $d\sigma$ and the ξ-axis is $\theta_1 = n\theta$, it is seen that a representation of the profile P can be obtained by

$$z = \int \frac{\tilde{q}}{q}\, e^{i\theta}\, d\sigma \tag{95}$$

The connection between the profiles P and $\sqcap$ can also be expressed by the formulas

$$\frac{R_P}{R_\sqcap} = n\,\frac{\tilde{q}}{q}, \qquad \frac{\theta}{\theta_1} = \frac{1}{n} \tag{96}$$

where R_p is the radius of curvature of the profile P at some point z and $R_\sqcap$ the radius of curvature of $\sqcap$ at the corresponding point ζ. Angle θ is known along $\sqcap$ and so is q, for $q^* = q_1^{1/n}$. If $\tilde{q}$ is known along $\sqcap$, then the profile P can be constructed graphically, using (95) or (96).

Solution of the Inverse Problem (Continued)

It has been shown in the preceding section that the construction of a compressible flow conjugate to a given incompressible flow depends upon the solution of a classical boundary value problem for a linear partial differential equation (equation (86)) or equation (87). The integration may be performed not in $E(\sqcap)$ but in a simpler domain, say in the domain exterior to the unit circle. For a conformal transformation of $E(\sqcap)$ into such a domain takes equations (86), (87) into equations of the same form and does not affect the auxiliary conditions. Nevertheless, the actual integration would be extremely laborious, especially in view of the fact that the coefficient T would be given either numerically or by a very complicated analytical expression. Further research is necessary in order to reduce the computational work to an acceptable minimum.

The physical interpretation of equations (85) given in the foregoing shows that these equations can be solved mechanically by G. I. Taylor's well-known method of the "electrolytic bath". (See reference 14.) It should be noted that, whereas Taylor applied his method in order to obtain a sequence of successive approximations to the solution of the direct problem (i.e., to the computation of the compressible flow past a given profile), in this case the method immediately furnishes the exact solution of the inverse problem.

For slow flows (i.e., for flows where the local Mach number is small) the following approximate method can be used.

The coefficient T in the equations (85) is equal to 1 for M = 0 and decreases very slowly as M increases to about 0.6, as seen from table I, where the polytropic relation with $\gamma = 1.405$ has been assumed (cf. also fig. 2).

Therefore, for low Mach numbers, the density $\tilde{\rho}$ of the fictitious flow in the ζ-plane (see preceding section) is almost constant so that q_i will be a good approximation to $\tilde{q}$. Replacing $\tilde{q}$ by $q_i = q^{*n}$ in the formulas (95), (96) yields

$$z = \int \frac{q^{*n}}{q} e^{i\theta} \, d\sigma \tag{97}$$

and

$$\frac{R_P}{R_\sqcap} = n \frac{q^{*n}}{q}, \quad \frac{\theta}{\theta_1} = \frac{1}{n} \tag{98}$$

Since $\frac{q}{q^{*n}}$ is known as a function of q_i and therefore known along $\sqcap$, the profile P can be immediately constructed.

From figure 2, where q and q^* are plotted as functions of M, it is seen that the profile distortion will be small for small values of M_∞ and for n close to 1.

This approximate method is based upon setting $T \equiv 1$. The same assumption is made in the Tchaplygin-Kármán-Tsien approximate method. However, there the equation of state is changed accordingly and the resulting differential equation is integrated rigorously; whereas here the rigorous equation is solved approximately. (However, see also reference 12, p. 348.)

Construction of a Flow around a Profile Similar to a Given Profile

Suppose it is desired to construct a compressible flow around a profile P similar to a given profile P_1 and having a prescribed stagnation point S. For the sake of definiteness it is assumed that P_1 possesses one sharp trailing edge.

Since the connection between n and M_∞ is not known a priori it is advisable to start by choosing a value for n and constructing a profile $\sqcap$ such that if P is obtained from $\sqcap$ by using (95) with this value of n a profile possessing only one sharp edge will result. Profile $\sqcap$ can be constructed by the method described in the preceding chapter.

Let $G(\zeta)$ be the potential of a flow around Π which has the velocity 1 at infinity. It is now necessary to find a value q^*_∞ such that if q^* is determined as

$$q^* = q^*\left(q^*_\infty n \left|\frac{dG}{d\zeta}\right|\right)$$

and T is determined as $T = T(q^*)$, the values of $\tilde{q}$ obtained by integrating equations (85) to (87) will lead to a closed profile P, when substituted in the relation (95). At the present state of the theory this can be achieved only by a trial-and-error method. It is convenient to start with the value of q^*_∞ given by (74) and then change this value so as to obtain a closed profile P. Since this involves the integration of the equations (85) to (87) for different functions T, the amount of computational work is rather considerable.

In the case of a circulation-free flow the situation is much simpler for q^*_∞ is independent of n and n may be taken as 1.

CONCLUDING REMARKS

A method is given to transform a two-dimensional incompressible flow around a closed profile into a subsonic compressible flow around another closed profile. The profile distortion is small for small values of the stream Mach number.

In the case of the actual equation of state the transformation depends upon the solution of a classical boundary value problem for a linear partial differential equation. An approximate method of solving this problem is indicated. Further theoretical work is required in order to establish the validity of the method in all cases and in order to reduce the amount of computational work.

It is believed that, after the solution of this inverse problem is completed, a way of solving the direct problem (computation of the flow around a given profile) will be open.

If Von Kármán's and Tsien's linearized equation of state is assumed, this transformation is carried out completely and in a closed form not only for flows without circulation (which was already done by Tsien) but for flows with circulation as well.

Concerning the applications of the linearized equation of state ($\gamma = -1$) the following may be said. This equation of state can be applied consistently. But, it also seems worthwhile to try to use the assumption $\gamma = -1$ only in order to obtain the values of the dimensionless speed q and to compute the resulting Mach number by means of the rigorous equation of state.

Other applications of solutions based upon setting $\gamma = -1$ also suggest themselves, for instance, to the solution of the exact equation of motion by successive approximations (by using the solution for $\gamma = -1$, instead of the solution for an incompressible flow, as the first approximation).

Numerical examples and a comparison with other methods will be given in a subsequent report.

Brown University,
Providence, R. I., April 29, 1944.

REFERENCES

1. Bers, Lipman, and Gelbart, Abe: On a Class of Differential Equations in Mechanics of Continua. Quarterly for Appl. Math., vol. 1, July 1943, pp. 168-188.

2. Bers, Lipman, and Gelbart, Abe: On a Class of Functions Defined by Partial Differential Equations. Trans. Am. Math. Soc., vol. 56, no. 1, July 1944, pp. 67-93.

3. Gelbart, Abe: On a Function-Theory Method for Obtaining Potential-Flow Patterns of a Compressible Fluid. NACA ARR No. 3G27, 1943.

4. Garrick, I. E., and Kaplan, Carl: On the Flow of a Compressible Fluid by the Hodograph Method. I - Unification and Extension of Present-Day Results. NACA ACR No. L4C24, 1944. (Classification changed to "Restricted," Oct. 1944.)

5. Molenbroek, P.: Über einige Bewegungen eines Gases mit Annahme eines Geschwindigkeitspotentials. Arch d. Math. und Phys., (2) vol. 9 (1890), pp. 157-186.

6. Tchaplygin, S. A.: On Gas Jets. Scientific Memoirs, Moscow Univ., Math. Phys. Sec., vol. 21, 1902, pp. 1-121. (Eng. trans., pub. by Brown Univ., 1944.) (Also NACA TM No. 1063, 1944)

7. Bergman, Stefan: A Formula for the Stream Function of Certain Flows. Proc. Nat. Acad. Sci., vol. 29, no. 9, Sept. 1943, pp. 276-281.

8. Temple, G., and Yarwood, J.: The Approximate Solution of the Hodograph Equations for Compressible Flow. RAS, S.M.E. Rep. 3201, June 1942.

9. Busemann, A.: Hodographenmethode der Gasdynamik. Z.f.a.M.M., vol. 12, April 1937, pp. 73-79.

10. Demtchenko, B.: Quelques problèmes d'hydrodynamique bidimensionelle des fluides compressibles. Pub. Sci. et Techn. du Ministère de l'Air, no. 144, 1939, pp. 1-40.

11. Tsien, Hsue-Shen: Two-Dimensional Subsonic Flow of Compressible Fluids. Jour. Aero. Sci., vol. 6, no. 10, Aug. 1939, pp. 399-407.

12. von Kármán, Th.: Compressibility Effects in Aerodynamics. Jour. Aero. Sci., vol. 8, no. 9, July 1941, pp. 337-356.

13. Theodorsen, Theodore: Theory of Wing Sections of Arbitrary Shapes. NACA Rep. No. 411, 1931.

14. Taylor, G. I., and Sharman, C. F.: A Mechanical Method for Solving Problems of Flow in Compressible Fluids. Proc. Roy. Soc., London, ser. A, vol. 121, 1928, p. 194.

APPENDIX

A. This section contains the proof of the following

<u>Lemma 1</u>: <u>Let</u>

$$\xi = \xi(x,y), \quad \eta = \eta(x,y) \tag{A1}$$

<u>be a transformation of a domain</u> D <u>in the</u> (x,y)-<u>plane into a domain</u> Δ <u>in the</u> (ξ,η)-<u>plane. Let</u> φ <u>and</u> ψ <u>be functions of</u> x <u>and</u> y <u>defined in</u> D. (<u>By virtue of the above transformation they may also be considered as functions of</u> ξ <u>and</u> η <u>defined in</u> Δ.) <u>If</u> φ <u>and</u> ψ <u>satisfy in</u> D <u>the differential equations</u>

$$\left.\begin{aligned} \frac{\partial\varphi}{\partial x} &= A\,\frac{\partial\psi}{\partial y} \\ \frac{\partial\varphi}{\partial y} &= -A\,\frac{\partial\psi}{\partial x} \end{aligned}\right\} \quad A = A(x,y) > 0 \tag{A2}$$

<u>and in</u> Δ <u>the differential equations</u>

$$\left.\begin{aligned} \frac{\partial\varphi}{\partial\xi} &= B\,\frac{\partial\psi}{\partial\eta} \\ \frac{\partial\varphi}{\partial\eta} &= -B\,\frac{\partial\psi}{\partial\xi} \end{aligned}\right\} \quad B = B(\xi,\eta) > 0 \tag{A3}$$

<u>then the transformation</u> (A1) <u>is conformal with respect to the following Riemann metric defined in</u> D:

$$dS^2 = e\,dx^2 + 2f\,dx\,dy + g\,dy^2 \tag{A4}$$

<u>where</u>

$$\left.\begin{aligned} e &= \cos^2\theta + \frac{B^2}{A^2}\sin^2\theta \\ f &= \left(1 - \frac{B^2}{A^2}\right)\sin\theta\cos\theta \\ g &= \sin^2\theta + \frac{B^2}{A^2}\cos^2\theta \end{aligned}\right\} \tag{A5}$$

and θ is the angle between the line ψ = constant and the x-axis:

$$\tan\theta = \frac{\partial\varphi}{\partial y}\bigg/\frac{\partial\varphi}{\partial x}$$

Geometrical proof. (Cf. fig. 7.)- Equations (A2) show that the lines φ = constant and ψ = constant subdivide the (x,y)-plane into infinitesimal rectangles of side ratio A. Similarly, equations (A3) express the fact that the lines φ = constant and ψ = constant subdivide the (ξ,η)-plane into infinitesimal rectangles of side ratio B. Therefore, in the neighborhood of some point (x,y) the mapping (A1) can be described as the product of a similarity transformation and a transformation which contracts all lengths in the direction of the line φ = constant in the ratio B/A. But a mapping conformal with respect to the metric (A4) is exactly such a mapping. For, let ds = (dx, dy) be a line element in the (x,y)-plane and let α be the angle between its direction and the x-direction. Then $dx = ds\cos\alpha$, $dy = ds\sin\alpha$, and $\chi = \pm(\theta - \alpha)$ is the angle between this line element and the line ψ = constant. A short computation shows that the non-Euclidean length of ds, as given by (A4), is equal to

$$dS = ds\sqrt{1 - (1 - B^2/A^2)\sin^2\chi}$$

Thus, for ds parallel to the line φ = constant, $dS = (B/A)ds$, and for ds parallel to the line ψ = constant, $dS = ds$.

Analytical proof.- Equations (A3) can be written in the form

$$\frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial\xi} + \frac{\partial\varphi}{\partial y}\frac{\partial y}{\partial\xi} = B\left(\frac{\partial\psi}{\partial x}\frac{\partial x}{\partial\eta} + \frac{\partial\psi}{\partial y}\frac{\partial y}{\partial\eta}\right)$$

$$\frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial\eta} + \frac{\partial\varphi}{\partial y}\frac{\partial y}{\partial\eta} = -B\left(\frac{\partial\psi}{\partial x}\frac{\partial x}{\partial\xi} + \frac{\partial\psi}{\partial y}\frac{\partial y}{\partial\xi}\right)$$

By use of the relations

$$\frac{\partial x}{\partial\xi} = J\frac{\partial\eta}{\partial y}, \quad \frac{\partial x}{\partial\eta} = -J\frac{\partial\xi}{\partial y}$$

$$\frac{\partial y}{\partial\xi} = -J\frac{\partial\eta}{\partial x}, \quad \frac{\partial y}{\partial\eta} = J\frac{\partial\xi}{\partial x}$$

$$J = \frac{\partial(x,y)}{\partial(\xi,\eta)}$$

this can be written in the form

$$\frac{\partial\varphi}{\partial x}\frac{\partial\eta}{\partial y} - \frac{\partial\varphi}{\partial y}\frac{\partial\eta}{\partial x} = -B\left(\frac{\partial\psi}{\partial x}\frac{\partial\xi}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\xi}{\partial x}\right)$$

$$\frac{\partial\varphi}{\partial x}\frac{\partial\xi}{\partial y} - \frac{\partial\varphi}{\partial y}\frac{\partial\xi}{\partial x} = B\left(\frac{\partial\psi}{\partial x}\frac{\partial\eta}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\eta}{\partial x}\right)$$

Introducing the values of $\partial\psi/\partial x$ and $\partial\psi/\partial y$ given by (A2) and using (A6) gives

$$\frac{B}{A}\cos\theta\frac{\partial\xi}{\partial x} + \frac{B}{A}\sin\theta\frac{\partial\xi}{\partial y} = -\sin\theta\frac{\partial\eta}{\partial x} + \cos\theta\frac{\partial\eta}{\partial y}$$

$$\sin\theta\frac{\partial\xi}{\partial x} - \cos\theta\frac{\partial\xi}{\partial y} = \frac{B}{A}\cos\theta\frac{\partial\eta}{\partial x} + \frac{B}{A}\sin\theta\frac{\partial\eta}{\partial y}$$

Solving for $\partial\xi/\partial x$ and $\partial\xi/\partial y$ gives

$$\frac{\partial\xi}{\partial x} = \frac{A}{B}\left\{\left(\frac{B^2}{A^2} - 1\right) \sin\theta \cos\theta \frac{\partial\eta}{\partial x} + \left(\frac{B^2}{A^2}\sin^2\theta + \cos^2\theta\right)\frac{\partial\eta}{\partial y}\right\}$$

$$\frac{\partial\xi}{\partial y} = -\frac{A}{B}\left\{\left(\sin^2\theta + \frac{B^2}{A^2}\cos^2\theta\right)\frac{\partial\eta}{\partial x} + \left(\frac{B^2}{A^2} - 1\right)\sin\theta\cos\theta\frac{\partial\eta}{\partial y}\right\}$$

If the notations given by (A5) are used, these equations can be written in the form

$$\frac{\partial\xi}{\partial x} = \frac{e\frac{\partial\eta}{\partial y} - f\frac{\partial\eta}{\partial x}}{+\sqrt{eg - f^2}}$$

$$\frac{\partial\xi}{\partial y} = -\frac{g\frac{\partial\eta}{\partial x} - f\frac{\partial\eta}{\partial y}}{+\sqrt{eg - f^2}}$$

But these are the well-known Beltrami equations which express the fact that the mapping (A1) is conformal with respect to the metric (A4).

<u>Remark</u>: The converse of the lemma is also true and can be proved similarly. A transformation (A1) conformal with respect to the metric (A4), (A5) takes φ and ψ satisfying (A2) into functions which satisfy equations (A3) in the (ξ,η)-plane.

B. This section contains the proof of lemma 2. <u>If</u> $E(P)$ <u>is mapped into</u> $E(\Pi)$ <u>by a transformation</u>

$$\xi = \xi(x,y), \qquad \eta = \eta(x,y)$$

<u>which is conformal with respect to a subsonic compressible flow around</u> P <u>and such that</u>

$$\xi^2 + \eta^2 \longrightarrow \infty$$

$$\frac{\partial \xi}{\partial x} \longrightarrow 1, \quad \frac{\partial \xi}{\partial y} \longrightarrow 0, \quad \frac{\partial \eta}{\partial x} \longrightarrow 0, \quad \frac{\partial \eta}{\partial y} \longrightarrow \sqrt{1 - M_\infty^2} \tag{B1}$$

$$\text{as} \quad x^2 + y^2 \longrightarrow \infty$$

then $u^* - iv^*$, considered as a function of $\zeta = \xi + i\eta$ has at the neighborhood of $\zeta = \infty$ the Laurent development

$$u^* - iv^* = q_\infty^* - \frac{i\Gamma_1}{2\pi} \frac{1}{\zeta} + \ldots \tag{B2}$$

where

$$\Gamma_1 = \sqrt{1 - M_\infty^2}\, \frac{q_\infty^*}{q_\infty}\, \Gamma \tag{B3}$$

Γ being the circulation of the flow around P.

The above transformation can be written in the form

$$\xi = A + \int_a^x \frac{\partial \xi(x', b)}{\partial x'}\, dx' + \int_b^y \frac{\partial \xi(x, y')}{\partial y'}\, dy'$$

$$= A + (x - a) + \int_a^x \left\{\frac{\partial \xi(x', b)}{\partial x'} - 1\right\} dx' + \int_b^y \frac{\partial \xi(x, y')}{\partial y'}\, dy'$$

$$\eta = B + \int_a^x \frac{\partial \eta(x', b)}{\partial x'}\, dx' + \int_b^y \frac{\partial \eta(x, y')}{\partial y'}\, dy'$$

$$= B + \int_a^x \frac{\partial \eta(x', b)}{\partial x'}\, dx' + N_\infty(y - b) + \int_b^y \left\{\frac{\partial \eta(x, y')}{\partial y'} - N_\infty\right\} dy'$$

where (a,b) is some point of $E(P)$,

$$A = \xi(a,b), \quad B = \eta(a,b)$$

and

$$N_\infty = \sqrt{1 - M_\infty^2}$$

If use is made of l'Hôpital's rule and (B1), it is seen that

$$\left.\begin{aligned} \xi &= x + o\left(\sqrt{x^2 + y^2}\right) \\ \eta &= N_\infty y + o\left(\sqrt{x^2 + y^2}\right) \end{aligned}\right\} x^2 + y^2 \longrightarrow \infty$$

Introducing

$$R^2 = x^2 + N_\infty^2 y^2$$

shows that

$$\left.\begin{aligned} \xi &= x + o(R) \\ \eta &= N_\infty y + o(R) \end{aligned}\right\} \qquad \text{(B4)}$$

Therefore:

$$\frac{1}{\zeta} = \frac{1}{x + iN_\infty y + o(R)} = \frac{1}{x + iN_\infty y} + o\left(\frac{1}{R}\right)$$

$$\frac{1}{\zeta^n} = O\left(\frac{1}{R^n}\right); \quad n > 1$$

Since $u^* - iv^*$ is a one-valued analytic function of ζ in $E(\Gamma)$ it possesses the Laurent development

$$u^* - iv^* = q_\infty^* + \frac{\alpha + i\beta}{\zeta} + \frac{\alpha_2 + i\beta_2}{\zeta^2} + \ldots \qquad \text{(B5)}$$

Hence, for sufficiently large values of $x^2 + y^2$,

$$u^* - iv^* = q_\infty^* + \frac{\alpha + i\beta}{x + iN_\infty y} + o\left(\frac{1}{R}\right) \qquad \text{(B6)}$$

so that

$$u^* = q^*_\infty + \frac{\alpha x + \beta N_\infty y}{x^2 + N_\infty^2 y^2} + o\left(\frac{1}{R}\right) \tag{B7}$$

$$v^* = \frac{\alpha N_\infty y - \beta x}{x^2 + N_\infty^2 y^2} + o\left(\frac{1}{R}\right) \tag{B8}$$

$$\begin{aligned} q^{*2} &= u^{*2} + v^{*2} \\ &= q_\infty^{*2} + 2q^*_\infty \frac{\alpha x + \beta N_\infty y}{x^2 + N_\infty^2 y^2} + o\left(\frac{1}{R}\right) \end{aligned} \tag{B9}$$

$$q^* = q^*_\infty + \frac{\alpha x + \beta N_\infty y}{x^2 + N_\infty^2 y^2} + o\left(\frac{1}{R}\right) \tag{B10}$$

Next, q/q^* is an analytic function of q^*, so that

$$\frac{q}{q^*} = \frac{q_\infty}{q^*_\infty} + A_\infty(q^* - q^*_\infty) + O(|q^* - q^*|^2) \tag{B11}$$

$$|q^* - q^*|^2 \longrightarrow 0$$

where

$$\begin{aligned} A_\infty &= \frac{d}{dq^*}\left(\frac{q}{q^*}\right)\Bigg|_{q^* = q^*_\infty} \\ &= \frac{1}{q^{*2}}\left\{q^* \frac{dq}{dq^*} - q\right\}\Bigg|_{q^* = q^*_\infty} \\ &= \frac{q_\infty}{q_\infty^{*2}}\left(\frac{1}{N_\infty} - 1\right) \end{aligned} \tag{B12}$$

since

$$\frac{dq^*}{dq} = \frac{d}{dq} \exp \int \frac{\sqrt{1 - M^2}}{q} dq = \frac{q^*}{q}\sqrt{1 - M^2}$$

(Cf. the definition of the conjugate speed q^*.)

By (B11) and (B10)

$$\frac{q}{q^*} = \frac{q_\infty}{q_\infty^*} + A_\infty \frac{\alpha x + \beta N_\infty y}{x^2 + N_\infty^2 y^2} + o\left(\frac{1}{R}\right)$$

Therefore

$$u = \frac{q}{q^*} u^* = q_\infty + \frac{q}{q_\infty^*} \frac{1}{N_\infty} \frac{\alpha x + \beta N_\infty y}{x^2 + N_\infty^2 y^2} + o\left(\frac{1}{R}\right) \tag{B13}$$

$$v = \frac{q}{q^*} v^* = \frac{q}{q_\infty^*} \frac{\alpha N_\infty y - \beta x}{x^2 + N_\infty^2 y^2} + o\left(\frac{1}{R}\right) \tag{B14}$$

The symbol ρ is an analytic function of q and therefore also of q^*. Hence

$$\rho = \rho_\infty + B_\infty(q^* - q_\infty^*) + O(|q^* - q^*_\infty|^2) \tag{B15}$$
$$|q^* - q_\infty^*| \longrightarrow 0$$

where

$$B_\infty = \left.\frac{d\rho}{dq^*}\right|_{q^* = q_\infty^*} = -\frac{\rho_\infty}{q_\infty^*} \frac{M_\infty^2}{N_\infty} \tag{B16}$$

since, by Bernoulli's equation,

$$\frac{d\rho}{dq} = -\frac{\rho}{q} M^2$$

By (B15) and (B10)

$$\rho = \rho_\infty + B_\infty \frac{\alpha x + \beta N_\infty y}{x^2 + N_\infty^2 y^2} + o\left(\frac{1}{R}\right) \tag{B17}$$

so that, by (B13), (B14), and (B17),

$$\rho u = \rho_\infty q_\infty + \rho_\infty \frac{q_\infty}{q_\infty^*} N_\infty \frac{\alpha x + \beta N_\infty y}{x^2 + N_\infty^2 y^2} + o\left(\frac{1}{R}\right) \tag{B18}$$

$$\rho v = \rho_\infty \frac{q_\infty}{q_\infty^*} \frac{\alpha N_\infty y - \beta x}{x^2 + N_\infty^2 y^2} + o\left(\frac{1}{R}\right) \tag{B19}$$

Now, let C be any simple closed curve containing the profile P in its interior. Then

$$\Gamma = \int_C u\, dx + v\, dy \tag{B20}$$

$$0 = \int_C \rho v\, dx - \rho u\, dy \tag{B21}$$

In particular, it is possible to take for C the ellipse C_R with the semi-axis

$$a = R, \quad b = \frac{R}{N_\infty}$$

the equation of which is

$$x^2 + N_\infty^2 y^2 = R^2$$

For sufficiently large values of R the developments for u, v, ρu, ρv previously obtained may be introduced under the integral signs in (B20), (B21). Then, using Green's theorem and denoting the interior of C_R by E_R results in

$$\Gamma = \int_{C_R} q_\infty\, dx + \frac{1}{R^2}\int_{C_R} \left\{\frac{q_\infty}{q_\infty^*}\frac{1}{N_\infty}(\alpha x + \beta N_\infty y)\, dx + \frac{q_\infty}{q_\infty^*}(\alpha N_\infty y - \beta x)\, dy\right\} + o(1)$$

$$= -\frac{2}{R^2}\frac{q_\infty}{q_\infty^*}\beta \iint_{E_R} dx\, dy + o(1)$$

$$= -\frac{2\beta}{N_\infty}\frac{q_\infty}{q_\infty^*}\pi + o(1)$$

$$0 = -\int_{C_R} \rho_\infty q_\infty \, dy + \frac{1}{R^2}\int_{C_R} \left\{ \rho_\infty \frac{q_\infty}{q_\infty^*}(\alpha N_\infty y - \beta x)dx \right.$$

$$\left. - \rho_\infty \frac{q_\infty}{q_\infty^*} N_\infty (\alpha x + \beta N_\infty y)dy \right\} + o(1)$$

$$= -\frac{\rho_\infty}{R^2}\frac{q_\infty}{q_\infty^*} 2\alpha N_\infty \iint_{E_R} dx\, dy + o(1)$$

$$= -2\alpha\rho_\infty \frac{q_\infty}{q_\infty^*}\pi + o(1)$$

Letting $R \longrightarrow \infty$ yields

$$\Gamma = -\frac{2\pi\beta}{N_\infty}\frac{q_\infty}{q_\infty^*}, \quad 0 = -2\pi\alpha\rho_\infty \frac{q_\infty}{q_\infty^*}$$

whence

$$\beta = -\frac{N_\infty \Gamma}{2\pi}\frac{q_\infty^*}{q_\infty}, \quad \alpha = 0 \tag{B22}$$

If this is substituted into (B5), it can be seen that (B2), (B3) is verified.

C. This section is devoted to the proof of the properties of the mapping function (70) which have been announced and used in deriving the parametric representation of subsonic flows with $\gamma = -1$.

First of all, this function maps $\sqcap$ into a <u>closed profile</u> P and is <u>one-valued</u> in $E(\sqcap)$. In fact, let J be a closed curve around $\sqcap$ and d the increase in z as ζ goes once around J (J may coincide with $\sqcap$). For sufficiently large values of $|\zeta|$, $G'(\zeta)$ has the Laurent development

$$G'(\zeta) = 1 + \frac{A_1}{i\zeta} + \frac{A_2}{\zeta^2} + \ldots, \quad A_1 \text{ real}$$

Therefore

$$d = C\left\{\int_J G'(\zeta)^{1-1/n}\, d\zeta - \mu^2 \int_J G'(\zeta)^{1+1/n}\, d\zeta\right\}$$

$$= 2\pi A_1\; C\left\{(1 - 1/n) - \mu^2\,(1 + 1/n)\right\}$$

If $A_1 = 0$, $d = 0$. If $A_1 \neq 0$, $G(\zeta)$ is not one-valued, so that $n = 1/\sqrt{1 - M_\infty^2}$ and therefore

$$(1 - 1/n) - \mu^2\,(1 + 1/n) = 0$$

so that $d = 0$.

Next, let $dz = ds\,e^{i\theta}$ be the complex line element on P, $d\zeta = d\sigma\, e^{i\theta_i}$ the corresponding complex line element on $\sqcap$. On $\sqcap$

$$G'(\zeta) = |G'|\;e^{-i\theta_i} \tag{C1}$$

so that by (70)

$$dz = C\,|G'|^{1-1/n}\,(1 - \mu^2\,|G'|^{2/n})\;e^{i\theta_i/n}\,d\sigma \tag{C2}$$

and therefore

$$\frac{ds}{d\sigma} = C\,|G'|^{1-1/n}\,(1 - \mu^2\,|G'|^{2/n})$$

By virtue of (69) $ds/d\sigma$ cannot vanish except at the two stagnation points where $G' = 0$, and is positive elsewhere. Hence the <u>mapping of</u> $\sqcap$ <u>into</u> P <u>is one-to-one</u>.

Furthermore, it is easily seen that z is <u>finite</u> for all finite values of ζ and $z = \infty$ for $\zeta = \infty$.

Finally, the <u>Jacobian</u>

$$\frac{\partial(x,\,y)}{\partial(\xi,\,\eta)}$$

<u>does not vanish in</u> $E(\sqcap)$. For

$$\left.\begin{aligned}\frac{\partial z}{\partial \xi} &= \frac{\partial z}{\partial \zeta} + \frac{\partial z}{\partial \bar{\zeta}} = C\left(G'^{1-1/n} - \mu^2\, \overline{G'^{1-1/n}}\right)\\ \frac{\partial z}{\partial \eta} &= i\left(\frac{\partial z}{\partial \zeta} - \frac{\partial z}{\partial \bar{\zeta}}\right) = iC\left(G'^{1-1/n} + \mu^2\, \overline{G'^{1+1/n}}\right)\end{aligned}\right\} \quad (C3)$$

And

$$\frac{\partial(x,\ y)}{\partial(\xi,\ \eta)} = \operatorname{Im}\left(\frac{\partial \bar{z}}{\partial \xi}\,\frac{\partial z}{\partial \eta}\right) = C^2\,|G'|^{2-2/n}\left(1 - \mu^4\,|G'|^{4/n}\right)$$

The expression on the right-hand side is not zero, for it is always assumed that there are no stagnation points within the flow and $(1 - \mu^4\,|G'|^{4/n}) > 0$ by virtue of (69).

From the preceding results it follows that (70) <u>maps</u> $E(\sqcap)$ <u>into a simply connected domain</u> $E(P)$ <u>containing the point infinity and bounded by a closed curve</u> P.

From (C3) and (65) it also follows that as $\xi \longrightarrow \infty$

$$\frac{\partial x}{\partial \xi} \longrightarrow C(1 - \mu^2), \quad \frac{\partial x}{\partial \eta} \longrightarrow 0, \quad \frac{\partial y}{\partial \xi} \longrightarrow 0, \quad \frac{\partial y}{\partial \eta} \longrightarrow C(1 + \mu^2) \quad (C4)$$

Finally, if $d\zeta = i\,d\sigma e^{i\theta_1}$ is a complex line element situated at a point of $\sqcap$ and normal to $\sqcap$, the corresponding line element dz is, by (70) and (C1) given by

$$dz = iC\,|G'|^{1-1/n}\left(1 + \mu^2\,|G'|^{2/n}\right) e^{i\theta_1/n}\, d\sigma$$

By comparison of this with (C2) it is seen that <u>the direction normal to</u> $\sqcap$ <u>is taken into the direction normal to</u> P.

TABLE I

The distorted speed (q^*), the Mach number (M), the dimensionless density (ρ), and the coefficient of the symmetrized hodograph equation (T) as functions of the dimensionless speed q. ($\gamma = 1.405$; $q_s = .911976$; only the mantissas of log q^* are given.)

q	$\log_{10} q^*$	q^*	M	ρ	T
.01	160453	.014169	.01000	.99995	1.00000
.02	461434	.028936	.02000	.99980	1.00000
.03	637472	.043398	.03000	.99955	1.00000
.04	762335	.057854	.04001	.99920	1.00000
.05	859146	.072301	.05001	.99875	1.00000
.06	938209	.086738	.06002	.99820	1.00000
.07	005014	.10116	.07003	.99755	.99999
.08	062842	.11557	.08005	.99680	.99999
.09	113609	.12996	.09007	.99595	.99998
.10	159359	.14433	.10010	.99501	.99997
.11	200533	.15868	.11014	.99396	.99996
.12	238060	.17301	.12018	.99282	.99994
.13	272549	.18730	.13022	.99157	.99991
.14	304438	.20158	.14028	.99023	.99988
.15	334063	.21582	.15034	.98879	.99984
.16	361771	.23002	.16042	.98725	.99980
.17	387738	.24420	.17050	.98561	.99974
.18	412176	.25833	.18059	.98388	.99967
.19	435249	.27243	.19070	.98205	.99959
.20	457094	.28648	.20081	.98012	.99950
.21	477830	.30049	.21094	.97809	.99939
.22	497557	.31445	.22108	.97597	.99926
.23	516362	.32837	.23124	.97376	.99911
.24	534321	.34223	.24141	.97145	.99895
.25	551503	.35604	.25160	.96904	.99875
.26	567966	.36980	.26180	.96654	.99855
.27	583762	.38350	.27202	.96394	.99829
.28	598938	.39713	.28225	.96126	.99801
.29	613533	.41071	.29250	.95847	.99769
.30	627590	.42422	.30277	.95560	.99735

TABLE I. (Continued)

q	$\log_{10} q^*$	q^*	M	ρ	T
.31	641139	.43766	.31306	.95265	.99695
.32	654211	.45104	.32337	.94958	.99652
.33	666833	.46434	.33370	.94643	.99604
.34	679031	.47756	.34405	.94319	.99551
.35	690827	.49071	.35442	.93986	.99492
.36	702243	.50378	.36482	.93644	.99427
.37	713298	.51677	.37524	.93294	.99356
.38	724009	.52967	.38568	.92934	.99278
.39	734392	.54249	.39615	.92566	.99192
.40	744463	.55522	.40664	.92189	.99099
.41	754235	.56785	.41716	.91804	.98997
.42	763721	.58039	.42771	.91410	.98886
.43	772932	.59283	.43828	.91008	.98765
.44	781881	.60517	.44889	.90597	.98633
.45	790576	.61741	.45952	.90178	.98490
.46	799027	.62955	.47018	.89750	.98336
.47	807244	.64157	.48088	.89315	.98168
.48	815234	.65348	.49161	.88872	.97986
.49	823004	.66528	.50236	.88420	.97789
.50	830563	.67696	.51316	.87961	.97577
.51	837916	.68852	.52399	.87494	.97347
.52	845071	.69996	.53485	.87019	.97099
.53	852032	.71127	.54575	.86537	.96832
.54	858805	.72245	.55669	.86047	.96545
.55	865396	.73349	.56766	.85549	.96233
.56	871808	.74440	.57868	.85044	.95898
.57	878047	.75517	.58973	.84532	.95538
.58	884116	.76580	.60083	.84013	.95150
.59	890019	.77628	.61196	.83486	.94733
.60	895761	.78661	.62314	.82953	.94284
.61	901343	.79679	.63437	.82412	.93800
.62	906769	.80681	.64564	.81865	.93280
.63	912042	.81666	.65695	.81312	.92721
.64	917164	.82635	.66832	.80751	.92119
.65	922138	.83587	.67973	.80185	.91472
.66	926966	.84521	.69119	.79612	.90775
.67	931649	.85438	.70270	.79032	.90025
.68	936190	.86336	.71426	.78447	.89217
.69	940590	.87215	.72588	.77855	.88347
.70	944849	.88074	.73755	.77258	.87408

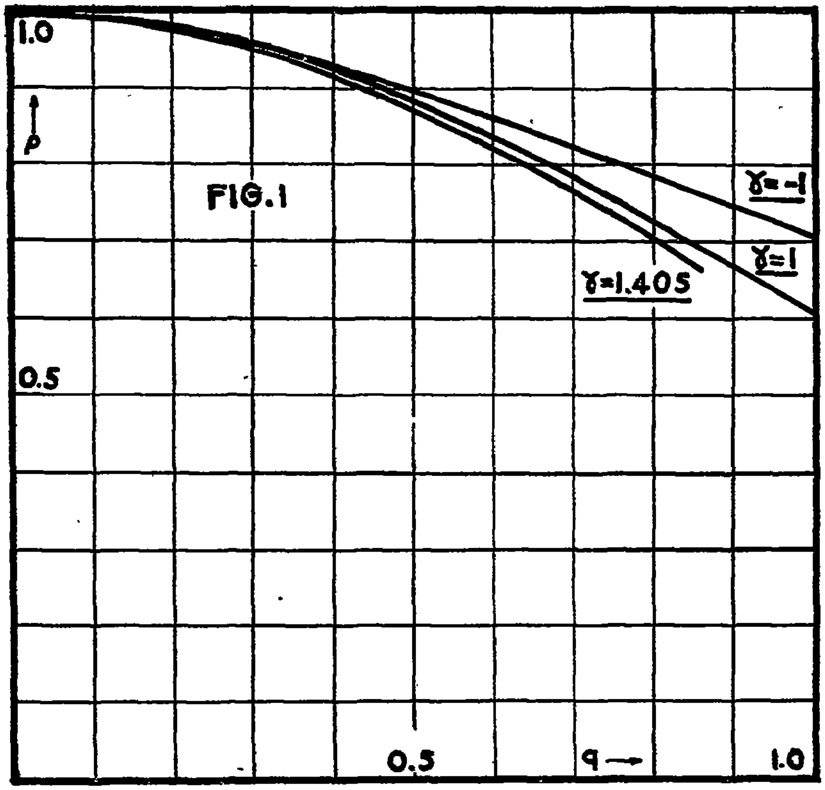
TABLE I. (Concluded)

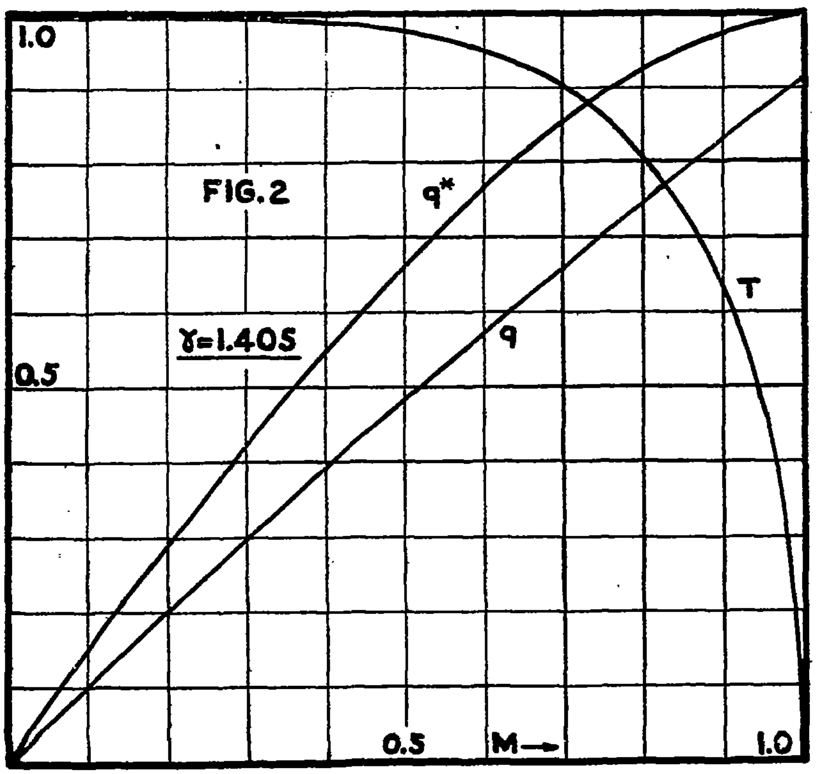
q	$\log_{10} q^*$	q^*	M	ρ	T
.71	948969	.88914	.74927	.76654	.86396
.72	952951	.89733	.76105	.76045	.85303
.73	956795	.90531	.77289	.75431	.84121
.74	960501	.91306	.78479	.74811	.82844
.75	964070	.92060	.79675	.74186	.81459
.76	967500	.92790	.80877	.73555	.79957
.77	970791	.93496	.82085	.72920	.78325
.78	973942	.94176	.83300	.72279	.76547
.79	976951	.94831	.84521	.71634	.74606
.80	979817	.95459	.85749	.70984	.72481
.81	982536	.96059	.86984	.70329	.70146
.82	985106	.96629	.88226	.69670	.67570
.83	987521	.97168	.89475	.69006	.64713
.84	989777	.97674	.90732	.68339	.61524
.85	991866	.98145	.91990	.67667	.57934
.86	993780	.98578	.93267	.66991	.53846
.87	995505	.98970	.94547	.66312	.49118
.88	997025	.99317	.95834	.65628	.43520
.89	998313	.99612	.97130	.64942	.36626
.90	999326	.99845	.98434	.64252	.27434
.91	999957	.99990	.99747	.63558	.11190
q_s	000000	1.00000	1.00000	.63425	.00000

TABLE II

Constants entering in the computation of the conjugate compressible flow for $\gamma = -1$.

M_∞	$n = \sqrt{\frac{1}{1-M_\infty^2}}$	1/n	$\mu = q^*_\infty$	μ^2	$K = \mu^{-1}$	$K = \mu^{-n}$
.05	1.001	.999	.025	.001	39.98	40.16
.10	1.005	.995	.050	.003	19.95	20.26
.15	1.011	.989	.075	.006	13.26	13.65
.20	1.021	.980	.101	.010	9.90	10.38
.25	1.033	.968	.127	.016	7.87	8.43
.30	1.048	.954	.154	.024	6.51	7.13
.35	1.068	.937	.181	.033	5.53	6.21
.40	1.091	.917	.212	.045	4.72	5.53
.45	1.120	.893	.238	.057	4.21	5.00
.50	1.155	.866	.268	.072	3.73	4.58
.55	1.197	.835	.300	.090	3.34	4.23
.60	1.250	.800	.333	.111	3.30	3.95
.65	1.306	.760	.369	.136	2.71	3.71
.70	1.400	.714	.408	.167	2.45	3.50
.75	1.535	.651	.454	.206	2.20	3.36
.80	1.667	.600	.500	.250	2.00	3.17
.85	1.898	.527	.557	.310	1.80	3.04
.90			.627	.393	1.60	
.95			.724	.524	1.38	
1.00			1.000	1.000	1.00	


1.0
ρ
FIG.1
γ=-1
γ=1
γ=1.405
0.5
0.5
q
1.0


1.0
FIG.2
q*
T
γ=1.405
q
0.5
0.5
M
1.0

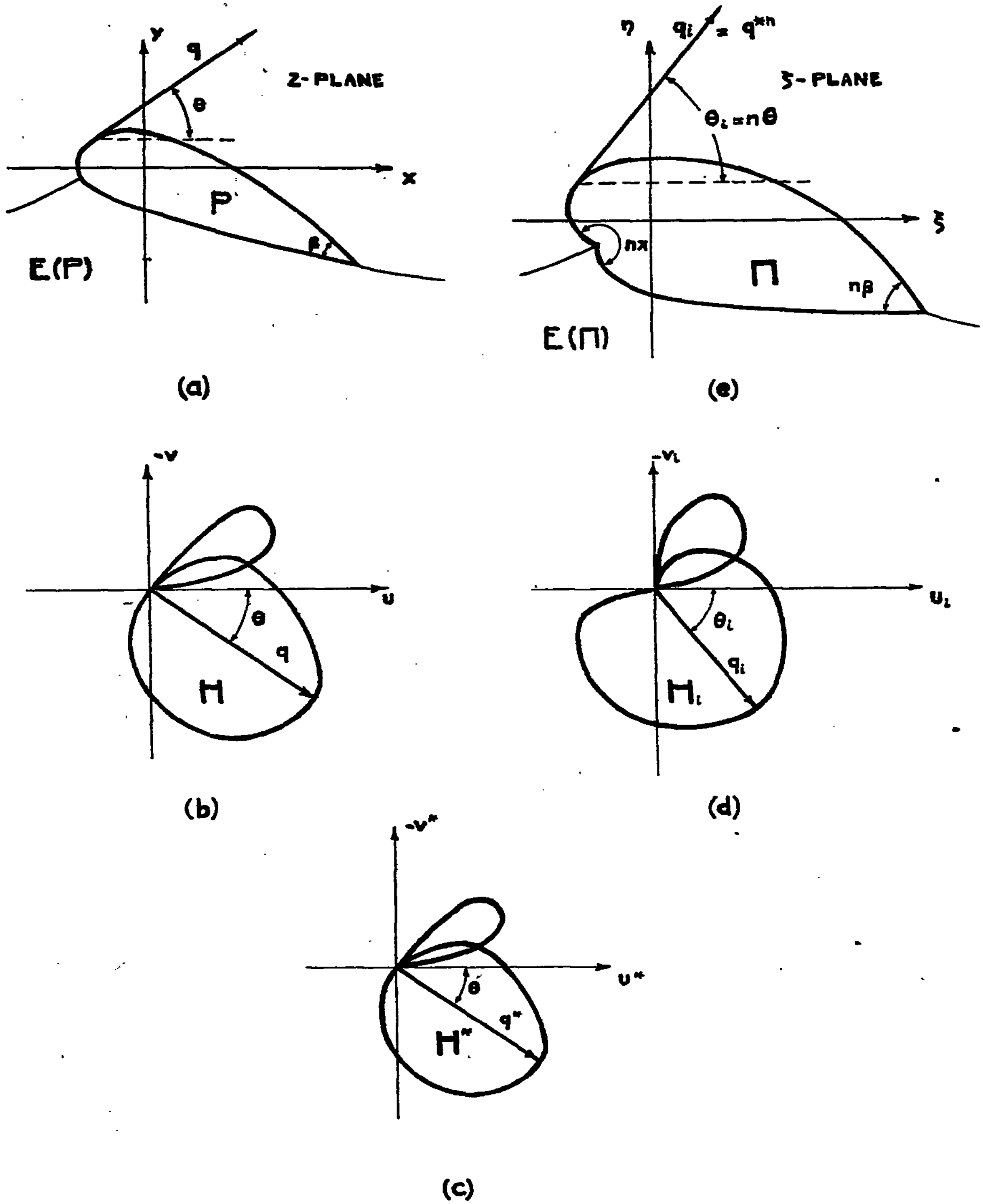
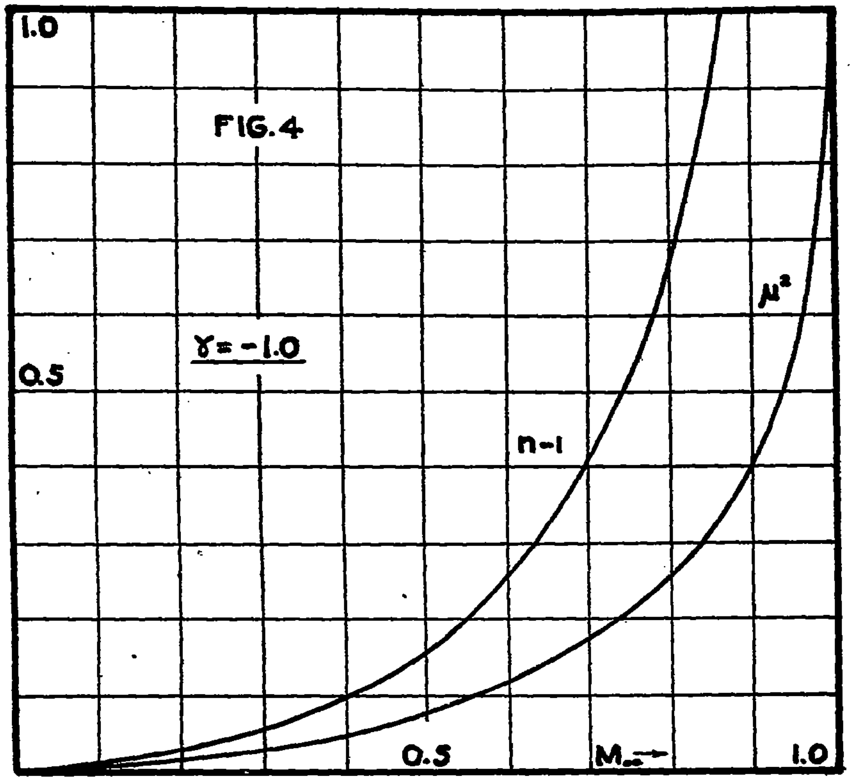
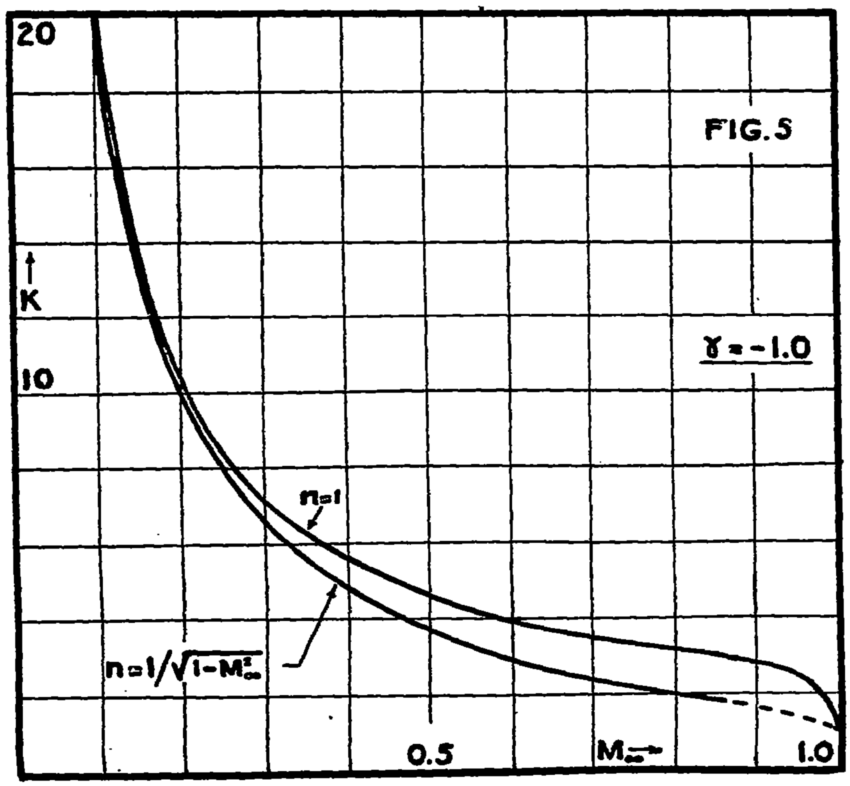

y
q
Z-PLANE
θ
x
P
β
E(P)
(a)
η
$q_i = q^{*n}$
ζ-PLANE
$\theta_i = n\theta$
ξ
nπ
Π
nβ
E(Π)
(e)
-v
u
θ
q
H
(b)
$-v_i$
u_i
θ_i
q_i
H_i
(d)
$-v^*$
u^*
θ^*
q^*
H^*
(c)

Fig.3



1.0
FIG.4
γ = -1.0
0.5
n-1
μ²
0.5
M∞→
1.0



20
FIG.5
↑K
10
γ = -1.0
n=1
n=1/√1-M∞²
0.5
M∞→
1.0

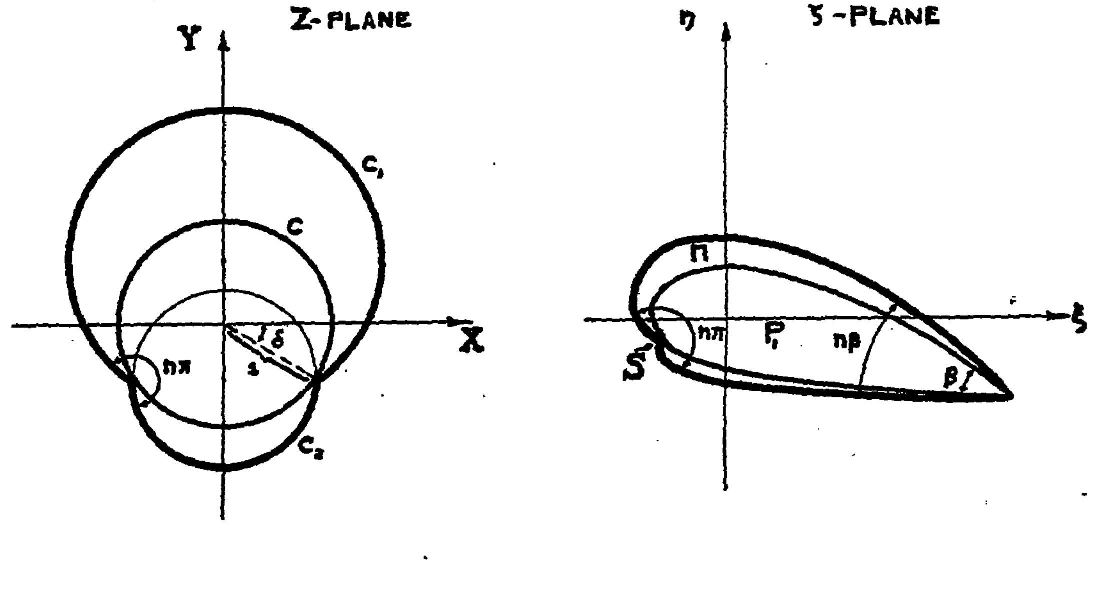

Z-PLANE
Y
X
C₁
C
C₂
nπ
δ
ζ-PLANE
η
ξ
nπ
nβ
β

Fig. 6

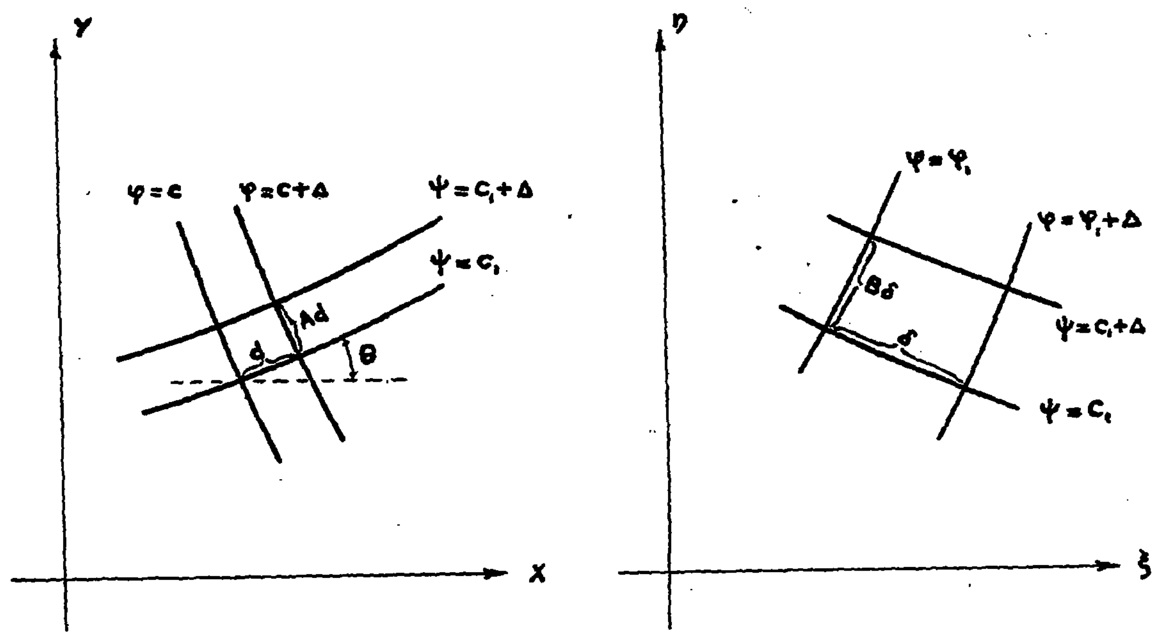

Y
X
φ=c
φ=c+Δ
ψ=c₁+Δ
ψ=c₁
Ad
d
θ
η
ξ
φ=φ₁
φ=φ₁+Δ
Bδ
δ
ψ=c₁+Δ
ψ=c₁

Fig. 7